Background on the Gan-Gross-Prasad Conjecture

Automorphic Project Seminar

David Schwein

University of Michigan

5 March 2020

Overview

Let G be a semisimple algebraic group (over a local field k, or sometimes a global field K).

Our goals are to (1) learn something about the words

tempered, generic, L-packets, A-packets, ...

and to (2) learn something about "restriction problems", one of which is the GGP conjecture.

This will help us to (3) understand Gan's talk next week.

Ramanujan-Petersson conjecture: statement

Conjecture (Ramanujan 1916)

Let

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n \qquad (q = \exp(2\pi i z))$$

be the discriminant modular form, a cusp form of weight 12. Then

$$|\tau(p)| \leq 2p^{(12-1)/2}.$$

The Ramanujan-Petersson conjecture (Petersson 1930) generalizes the conjecture to cover Maass forms as well.

Satake reformulated the Ramanujan-Petersson conjecture in the language of automorphic representations.

Ramanujan-Petersson conjecture: Satake's reformulation

Recall that every admissible irrep π of $G(\mathbb{A}_K)$ is a product of irreps π_v of the group $G(K_v)$:

$$\pi = \bigotimes_v{}' \pi_v.$$

π is automorphic if it is a subquotient of $L^2(G(K)\backslash G(\mathbb{A}_K))$, and cuspidal (automorphic) if it is a subrep of $L^2_{\text{cusp}}(G(K)\backslash G(\mathbb{A}_K))$.

Conjecture (Satake 1965)

The local components of a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ are tempered.

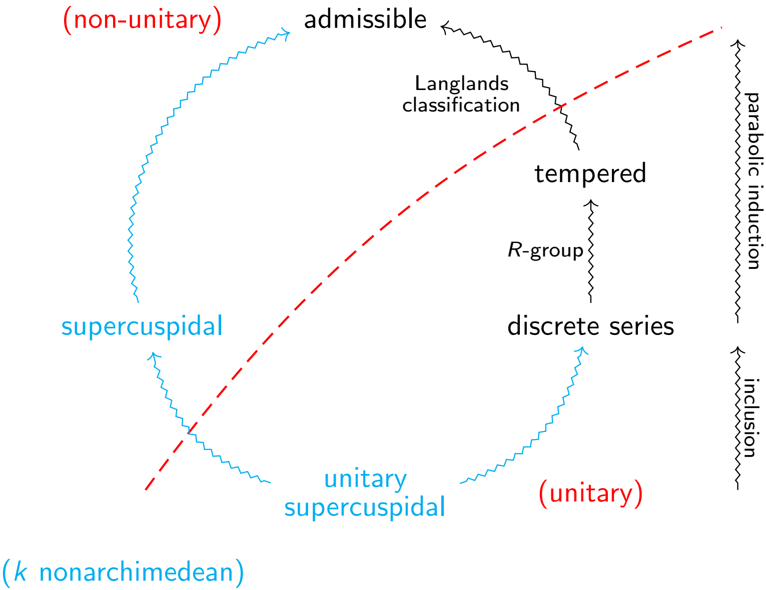
Tempered representations: definition

An irreducible admissible rep π of a semisimple group $G(k)$ is tempered if one of the following equivalent conditions holds.

1. Its character is a tempered distribution (extends to Schwartz).
2. Its matrix coefficients lie in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.
3. Its exponents are nonnegative (for every parabolic).
4. It is unitary and lies in the support of the Plancherel measure.
5. It is a subrep of a parabolic induction of a discrete series.

Tempered representations are also the building blocks of (irreducible) admissible representations, via parabolic induction.

Tempered representations: hierarchy



Howe and Piatetski-Shapiro's counterexample

Satake's conjecture has an obvious naive generalization.

Naive Conjecture (Wrong!)

For any G, the local components of a cuspidal automorphic representation of $G(\mathbb{A}_K)$ are tempered.

However, the naive generalization is false already for Sp_4!

Theorem (Howe and Piatetski-Shapiro 1977)

There is a cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_K)$ that is nontempered almost everywhere.

Questions:

1. Can the naive conjecture be salvaged?
2. What are the local components of automorphic reps?

Ramanujan-Petersson conjecture: salvage

Some of the local components of Howe and Piatetski-Shapiro's counterexample are θ_{10} reps constructed by Srinivasan (1968).

θ_{10} fails to admit a Whittaker model.

On the other hand, the local components of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ *do* have Whittaker models.

We can generalize Satake's conjecture by adding a Whittaker model hypothesis.

Whittaker models: generic characters

Assume that G is quasi-split, that is, has a Borel $B = TU$.

A character ψ of $U(k)$ is generic if it is nontrivial on every simple root group.

Whittaker datum: a pair $\mathfrak{w} = (B, \psi : U(k) \to \mathbb{C}^\times)$ with ψ generic.

Example

Let $G = \mathrm{SL}_n$ and let $\psi_0 : k \to \mathbb{C}^\times$ be a character. The character

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \dots \\ & 1 & a_{23} & \dots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \mapsto \psi_0\big(b_1 a_{12} + b_2 a_{23} + \cdots + b_{n-1} a_{n-1,n}\big)$$

is generic if and only if each b_i is nonzero.

Whittaker models: definition

Let $\mathfrak{w} = (B, \psi : U(k) \to \mathbb{C}^\times)$ be a Whittaker datum.

A ($\mathfrak{w}$-)Whittaker model of an admissible irrep (π, V) of $G(k)$ is the image of an injective intertwiner

$$\pi \hookrightarrow \mathrm{Ind}_{U(k)}^{G(k)} \psi \qquad \left(\begin{array}{l}\text{Gelfand-Graev}\\ \text{representation}\end{array}\right).$$

Alternatively, a ($\mathfrak{w}$-)Whittaker functional is a (nonzero) continuous linear functional $\lambda : V \to \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \psi(u)\lambda(v), \qquad u \in U(k), v \in V.$$

The two notions correspond under Frobenius reciprocity:

$$\mathrm{Hom}_{U(k)}(\pi|_{U(k)}, \psi) = \mathrm{Hom}_{G(k)}\left(\pi, \mathrm{Ind}_{U(k)}^{G(k)} \psi\right).$$

Whittaker models: existence and uniqueness

Broadly speaking, Whittaker models are useful because they realize representations concretely, in a space of functions on the group.

Theorem (Shalika 1974)

A $\mathfrak{w}$-Whittaker model of π is unique, if it exists.

π is ($\mathfrak{w}$-)generic if it admits a ($\mathfrak{w}$-)Whittaker model.

Theorem

Every local component of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ is generic.

On the other hand, the irrep θ_{10} of $\mathrm{Sp}_4(k)$ is not generic.

Whittaker models: applications

Conjecture (Generalized Ramanujan)

If a cuspidal automorphic representation is globally generic then each of its local components is tempered.

The Langlands-Shahidi method constructs (analytically!) L-functions of certain generic cuspidal automorphic representations.

Generic representations are expected to pin down the parameterization of tempered L-packets.

On to A-packets

Recall our two questions, the first of which is now answered:

1. Can the naive generalization of the conjecture be salvaged?
2. What are the local components of automorphic reps?

Arthur's conjectural answer to the second question uses the local Langlands correspondence.

So we'll learn about L-packets, then A-packets.

For simplicity, assume $k \neq \mathbb{R}, \mathbb{C}$.

L-parameters: source (Weil-Deligne group)

An *L-parameter* is a generalization of a complex Galois representation $\Gamma_k \to \mathrm{GL}_n(\mathbb{C})$.

Galois group: $\Gamma_k = I_k \rtimes \widehat{\mathbb{Z}}$

Weil group: $W_k = I_k \rtimes \mathbb{Z} \xrightarrow{|\cdot|} q^{\mathbb{Z}} \subset \mathbb{R}^{\times}$

Weil-Deligne group: $WD_k = W_k \times \overbrace{\mathrm{SL}_2(\mathbb{C})}^{\text{Deligne } \mathrm{SL}_2}$ (variant: $W_k \ltimes \mathbb{C}$)

Groups: $WD_k \twoheadrightarrow W_k \xhookrightarrow{\text{dense}} \Gamma_k$

Reps: Weil-Deligne $\hookleftarrow$ Weil $\hookleftarrow$ Galois

L-parameters: Weil-Deligne representations

A(n admissible) Weil-Deligne rep $WD_k = W_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$ has the form

$$V = \bigoplus_{d \geq 0} V_d \boxtimes \mathrm{Sym}^d$$

where V_d is a semisimple rep of W_k and Sym^d is the unique irrep of $\mathrm{SL}_2(\mathbb{C})$ of dim $d + 1$.

Say V admits a WD_k-invariant bilinear form B, that is, an isomorphism $f : V \to V^\vee$. Then $f^\vee = \pm f : V^{\vee\vee} = V \to V^\vee$.

- V is orthogonal $\iff f = +f$.
- V is symplectic $\iff f = -f$.

L-parameters: target (L-group) and definition

Langlands dual group: $G \mapsto \widehat{G}$.

L-group of G: ${}^L G = \widehat{G}(\mathbb{C}) \rtimes \Gamma_k$.

For G split, ${}^L G = \widehat{G}(\mathbb{C}) \times \Gamma_k$.

An L-parameter for GL_n is a Weil-Deligne representation.

G	$\widehat{G}$
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

L-parameter for G: a continuous hom $\varphi : WD_k \to {}^L G$ such that

1. φ commutes with the maps to Γ_k,
2. $\varphi(W_k)$ is semisimple, and
3. $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. (variant: $\varphi(\mathbb{C})$ is unipotent)

Two L-parameters are equivalent if they are $\widehat{G}(\mathbb{C})$-conjugate.

L-parameters: classical groups

$$WD_k \to {}^L G \quad \longleftrightarrow \quad WD_k \curvearrowright V$$

G	V	$\dim V$	
SL_n	projective	n	
Sp_{2n}	orthogonal	$2n+1$	$(\det V = 1)$
SO_{2n+1}	symplectic	$2n$	
SO_{2n}	orthogonal	$2n$	$(\det V = \operatorname{disc} V)$
U_{2n+1}	conjugate-orthogonal	$2n+1$	
U_{2n}	conjugate-symplectic	$2n$	

In the orthogonal case it can happen that inequivalent parameters produce isomorphic orthogonal reps.

L-packets: "definition"

Conjecture (Langlands)

There is a surjective map

$$\frac{\{\text{admissible irreps of } G(k)\}}{\text{isomorphism}} \longrightarrow \frac{\{L\text{-parameters } WD_k \to {}^L G\}}{\text{equivalence}}$$

satisfying many nice properties. The fibers of this map, called L-packets, are finite.

L-packets are singletons if G is a torus or GL_n (so that the LLC is a bijection), but are generally not singletons otherwise.

Write $\Pi(\varphi)$ for the L-packet of $\varphi : WD_k \to {}^L G$.

L-packets: properties

It is expected that properties of a parameter are reflected in properties of the representations in its L-packet.

Representations π	Parameters φ
unramified	$\varphi(I_k \times \mathrm{SL}_2(\mathbb{C})) = 1$
supercuspidal	(Aubert, Moussaoui, Solleveld 2017)
(essentially) discrete	centralizer of φ is finite
tempered	$\mathrm{im}_{\widehat{G}}(\varphi(W_k))$ is bounded
unitary	?? (ATLAS)

A tempered L-parameter ought to know the Plancherel measure of the representations in its L-packet (Hiraga, Ichino, Ikeda 2008).

L-packets: parameterization (tempered case)

Assume for simplicity that G is quasi-split.

Let S_φ be the centralizer of φ in $\widehat{G}(\mathbb{C})$.

Conjecture

Let φ be a tempered parameter.

1. *(Shahidi) For each Whittaker datum $\mathfrak{w}$, the L-packet $\Pi(\varphi)$ contains a unique $\mathfrak{w}$-generic representation $\pi_{\mathfrak{w}}$.*
2. *There is an injection (bijection if k is nonarchimedean)*

$$i_{\mathfrak{w}} : \Pi(\varphi) \to \mathrm{Irr}\big(\pi_0\big(S_\varphi / Z(\widehat{G})^\Gamma\big)\big)$$

   sending $\pi_{\mathfrak{w}}$ to triv.

A-parameters: motivation

Tempered L-packets and params are well understood conjecturally.

But not all arithmetically interesting reps are tempered, e.g. θ_{10}.

If we are interested in arithmetically interesting reps then

- tempered L-parameters describe too few reps but
- arbitrary L-parameters describe too many reps (e.g. the nonunitary ones).

Arthur proposed an intermediate parameter whose packets conjecturally capture the arithmetically interesting reps.

$$\begin{matrix}\text{tempered} \\ L\text{-parameters}\end{matrix} \hookrightarrow A\text{-parameters} \hookrightarrow \begin{matrix}\text{arbitrary} \\ L\text{-parameters}\end{matrix}$$

A-parameters: definitions

An A-parameter for G is a homomorphism

$$\psi : WD_k \times \underbrace{\mathrm{SL}_2(\mathbb{C})}_{\text{Arthur } \mathrm{SL}_2} \to {}^L G$$

such that $\psi|_{WD_k}$ is a tempered L-parameter and $\psi|_{\mathrm{SL}_2}$ is algebraic.

Equivalence of parameters is $\widehat{G}(\mathbb{C})$-conjugacy.

An A-parameter ψ gives rise to an L-parameter φ_ψ by

$$\varphi_\psi(w, g) = \psi\left(w, g, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right).$$

A-parameters: classical groups

An A-parameter for GL_n is an n-dimensional representation of $W_k \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$:

$$V = \bigoplus_{d,e \geq 0} V_{d,e} \boxtimes \mathrm{Sym}^d \boxtimes \mathrm{Sym}^e \,.$$

Here $V_{d,e}$ is a semisimple rep of W_k and Sym^d is the unique irrep of $\mathrm{SL}_2(\mathbb{C})$ of dim $d+1$.

There are also concrete descriptions for A-parameters of other classical groups, as for L-parameters.

(Local) A-packets

Conjecturally, one can attach to each A-parameter ψ a (multi)set $\Pi(\psi)$ of irreps of $G(k)$, the A-packet of ψ.

We expect that

- if G is quasi-split then $\Pi(\varphi_\psi) \subseteq \Pi(\psi)$,
- every local component of an automorphic representation lies in some A-packet, and
- every member of $\Pi(\psi)$ is unitary.

Warning: A-packets need not be disjoint!

Restriction problem: statement

Problem

Given a group G, a subgroup H, and a rep π of G, what is $\pi|_H$?

Often π is irreducible and it is enough to know the irreducible subreps (or quotients) of $\pi|_H$.

Problem

Given groups $H \subseteq G$, an irrep π of G, and an irrep ρ of H, what is

$$\dim \mathrm{Hom}_H(\pi|_H, \rho)?$$

Variant: a rep π of G is *H*-distinguished if $\pi^H \neq 0$. Then

$$\dim \mathrm{Hom}_H(\pi|_H, \rho) > 0 \iff \pi \boxtimes \rho^\vee \text{ is dist wrt } H \xrightarrow{\text{diag}} G \times H.$$

Restriction problem: unitary group

Highest weight theory shows that the irreps of (compact, real) U_n are in bijection with integer sequences

$$a = (a_1 \geq a_2 \geq \cdots \geq a_n).$$

Let V_a be the corresponding irrep.

For $b = (b_1 \geq \cdots \geq b_{n-1})$ let $d(a, b) = \dim \operatorname{Hom}_{U_{n-1}}(V_a|_{U_{n-1}}, V_b)$

Theorem

1. *$d(a, b) \leq 1$.* *(multiplicity at most one)*
2. *$d(a, b) = 1$ if and only if b interlaces a:*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq b_{n-1} \geq a_n.$$

Multiplicity (at most) one

The case of GL_n is understood through work of [Aizenbud, Gourevitch, Rallis, and Schiffmann] (nonarchimedean case) and [Aizenbud and Gourevitch] and [Sun and Zhu] (archimedean case).

Theorem

Let k be a characteristic zero local field and let π (resp. ρ) be an irreducible admissible representation of GL_n (resp. GL_{n-1}). Then

$$\dim \mathrm{Hom}_{\mathsf{GL}_n}(\pi|_{\mathsf{GL}_{n-1}}, \rho) \leq 1.$$

Multiplicity at most one is expected to hold for other classical groups, and this is proved in many cases.

The Gan-Gross-Prasad conjecture predicts exactly when the multiplicity is one.

Multiplicity one: generic representations of GL_n

The following result is a folk theorem whose proof was written down by Prasad.

Theorem

Let k be a p-adic field and let (π, V) (resp. (ρ, W)) be an irreducible admissible representation of GL_n (resp. GL_{n-1}). If π and ρ are generic then

$$\dim \mathrm{Hom}_{\mathrm{GL}_n}(\pi|_{\mathrm{GL}_{n-1}}, \rho) = 1.$$

The proof uses Jacquet, Piatetskii-Shapiro, and Shalika's theory of Rankin-Selberg convolutions to construct a nonzero, GL_{n-1}-invariant form $B : V \otimes W^{\vee} \to \mathbb{C}$.

Multiplicity one: proof sketch

Let $i : V \to \mathrm{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$ and $j : W \to \mathrm{Ind}_{U_{n-1}}^{\mathrm{GL}_{n-1}} \overline{\psi}_{n-1}$ be Whittaker models, where

$$\overline{\psi}_{n-1} \cdot \psi_n|_{U_{n-1}} = 1.$$

For $s \in \mathbb{C}$, let $B_s(v, w^\vee) = \displaystyle\int_{\mathrm{GL}_{n-1}/U_{n-1}} i_v(x \oplus e_{nn})\, j_{w^\vee}(x)\, |\det x|^s \, dx.$

JP-SS show that

- B_s converges if $\mathrm{Re}\, s \gg 0$,
- $s \mapsto B_s$ admits a meromorphic continuation to all of $\mathbb{C}$, and
- there are $v_0 \in V$ and $w_0^\vee \in W^\vee$ such that $B_0(v_0, w_0^\vee) = 1$.

Use B_0 (or possibly the residue at $s = 0$) to prove the theorem.

Thank you for your attention!