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# Exercises, Algebra I (Commutative Algebra) – Week 4

**Exercise 15.** (Scalar extension of Ext and Tor, 3 points)
Assume $A \to B$ is a ring homomorphism with $B$ flat over $A$. Let $M$ and $N$ be $A$-modules. Show that there exist natural isomorphisms $\mathrm{Tor}_i^A(M,N) \otimes_A B \cong \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B)$. (There was a mistake in the previous version of the Exercise)
Assume moreover that $A$ is Noetherian and $M$ is finitely generated. Show that there exist natural isomorphisms $\mathrm{Ext}^i(M,N) \otimes_A B \cong \mathrm{Ext}_B^i(M \otimes_A B, N \otimes_A B)$.

**Exercise 16.** (Properties of elements in polynomial rings, 4 points)
Consider the polynomial ring $A[x]$ for an arbitrary ring $A$ and let $0 \neq f = a_0 + a_1 x + \ldots a_n x^n \in A[x]$. Prove the following assertions

(i) Prove that if $a \in A$ is nilpotent and $b \in A^\times$ is a unit, $a + b$ is a unit.

(ii) $f$ is a unit if and only if $a_0$ is a unit and $a_i$, $i > 0$ are nilpotent (hint: if $g = \sum_{i=0}^d b_i x^i$ is an inverse of $f$, prove that $a_n^{k+1} b_{d-k} = 0$ for any $k \geq 0$).

(iii) $f$ is nilpotent if and only if all $a_i$ are nilpotent.

(iv) $f$ is a zero divisor if and only if there exists an $0 \neq a \in A$ with $af = 0$.

**Exercise 17.** (Short exact sequences, 1 point)
Assume $0 \longrightarrow M_1 \longrightarrow M_2 \xrightarrow{\pi} M_3 \longrightarrow 0$ is a short exact sequence of $A$-modules. Show that for any $A$-submodule $N_3 \subset M_3$ also $0 \longrightarrow M_1 \longrightarrow N_2 \xrightarrow{\pi} N_3 \longrightarrow 0$ is exact. Here, $N_2 := \pi^{-1}(N_3)$.

**Exercise 18.** (Examples of nilradicals, 4 points)
Describe the nilradical $\mathfrak{N}$ and the Jacobson radical $\mathfrak{R}$ for the following rings ($k$ denotes a field): $A = k[x]$; $A = k[[x]]$; $A = k[x]/(x^3)$; $A = \mathbb{Z}/(18)$.

**Exercise 19.** (Rings with one prime ideal, 2 points)
Let $A$ be a ring and $\mathfrak{N} \subset A$ its nilradical. Show that the following conditions are equivalent:

(i) $A$ has exactly one prime ideal.

(ii) Every element in $A$ is either a unit or nilpotent.

(iii) $A/\mathfrak{N}$ is a field.

**Exercise 20.** (Radical $\sqrt{\mathfrak{a}}$, 3 points)
Let $\mathfrak{a}, \mathfrak{b} \subset A$ ideals. Prove the following assertions:

(i) $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.

(ii) $\sqrt{\mathfrak{a}} = (1)$ if and only if $\mathfrak{a} = (1)$.

(iii) Assume $\mathfrak{a} \neq (1)$. Then $\mathfrak{a} = \sqrt{\mathfrak{a}}$ if and only if $\mathfrak{a}$ is an intersection of prime ideals

**Exercise 21.** (Faithfully flatness, 3 points)
An $A$-module $M$ is called *faithfully flat* if $M$ is flat and for all $A$-modules $N_1, N_2$ the natural map $\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(M \otimes N_1, M \otimes N_2)$ is injective. Show that every free module is faithfully flat. Recall that every projective module is flat. Is it also faithfully flat?

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Solutions to be handed in before Monday May 4, 4pm.