Prof. Dr. Daniel Huybrechts
Dr. René Mboro Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 10

Exercise 49. (Associated primes, 4 points)
Let A be a ring and M an A-module. A prime ideal $\mathfrak{p} \subset A$ is *associated* with M if there exists an $m \in M$ with $\mathrm{Ann}(m) = \mathfrak{p}$ or, equivalently, if there exists an injection $A/\mathfrak{p} \hookrightarrow M$. The set of associated prime ideals is denoted $\mathrm{Ass}(M)$.

(i) Let N be a submodule of M. Prove that $\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$.

(ii) Show that for all $\mathfrak{p} \in \mathrm{Ass}(M)$ one has $M_{\mathfrak{p}} \neq 0$, i.e. $\mathfrak{p} \in \mathrm{Supp}(M)$.

(iii) Assuming A to be Noetherian, prove that the natural map $M \to \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is injective.
Hint: If the kernel is not trivial, find an element in the kernel whose annihilator is prime.

Exercise 50. (Discrete valuation rings (or not), 6 points)
Decide which of the following rings are discrete valuation rings:

$$\mathbb{Z};\ k[[x]];\ k[x]_x;\ k[x^2, x^3] \subset k[x];\ \mathbb{F}_3[x, y]/(x^2 - y).$$

Assume $\nu\colon K^* \to \mathbb{Z}$ is a valuation (i.e. a map satisfying Lemma 13.4 (i) and (ii)) with associated valuation ring A. Is A a discrete valuation ring?

Exercise 51. (Rings that are not Dedekind rings, 5 points)
(i) Show that $A \coloneqq k[x_1, x_2]$ is not a Dedekind ring by describing a non-zero fractional ideal that is not invertible.

(ii) We know that $A = k[x_1, x_2]/(x_2^2 - x_1^3)$ is not normal and hence not a Dedekind ring. Find a non-zero fractional ideal that is not invertible.
Hint: One can try to find an equation satisfied by the ideal $(\overline{x_1}, \overline{x_2})$.

Exercise 52. (Absolute values, 4 points)
An *absolute value* on an integral domain A is a map $|\cdot|\colon A \to \mathbb{R}$ such that for all $a, b \in A$:

$$(1)\ |a| \geq 0,\ (2)\ |a| = 0 \text{ if and only if } a = 0,\ (3)\ |ab| = |a| \cdot |b|, \text{ and } (4)\ |a + b| \leq |a| + |b|.$$

(i) Check that any absolute value on A extends to an absolute value on its fraction field $Q(A)$ (define $|a/b| = |a|/|b|$).

(ii) Suppose (4) is replaced by the stronger requirement $|a + b| \leq \max\{|a|, |b|\}$. Show that then for any $\alpha > 1$ the map $\nu\colon Q(A)^* \to \mathbb{R}$, $x \mapsto -\log_\alpha |x|$ is a valuation.

(iii) What goes wrong for $\mathbb{C}^* \to \mathbb{R}$, $x \mapsto -\log_\alpha |x|$?

(iv) Determine an absolute value giving rise to a valuation on $\mathbb{Q}$ whose valuation ring is $\mathbb{Z}_{(p)}$.

Solutions to be handed in before Tuesday June 22, 4pm.

Exercise 53. (Picard group, 6 points)
For any ring A one defines $\mathrm{Pic}(A)$ as the set of all isomorphism classes of finite projective A-modules M such that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

(i) Show that $\mathrm{Pic}(A)$ with $(M, N) \mapsto M \otimes_A N$ is an abelian group. (The *Picard group.*)

(ii) Show that for a Dedekind ring the map $\mathrm{Cl}(A) \to \mathrm{Pic}(A)$ that forgets the inclusion $M \subset K = Q(A)$ of a fractional ideal is an isomorphism.

Exercise 54. (Class number, 5 points)
Prove that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$ have class number one, i.e.

$$h_{-1} = 1 \text{ and } h_{\sqrt{-2}} = 1.$$

Hint: In each case, one can define an absolute value and imitate Euclidean algorithm.