FORMAL PROOF OF AN USEFUL INEQUALITY INVOLVING $(n-1)-$HAUSDORFF MEASURES OF LEVEL SETS

1. Motivation

Let Ω be a C^2-domain, and let

$$U := \{x \in \Omega \colon d(x, \partial\Omega) \sim 2^k, 2^{-j-1} < \frac{|y - b_x|}{d(x, \partial\Omega)} < 2^{-j+1}\},$$

where b_x is the nearest point in the boundary to x (exists, is unique and Lipschitz for C^2-domains if x is sufficiently close to the boundary). Let also

$$g_y(x) := |y - x| - d(x, \partial\Omega).$$

This function is Lipschitz in Ω in general, but also C^1 with Lipschitz gradient in $U \backslash B(y, \delta)$, as its gradient is explicitly given by

$$\nabla_x g_y(x) = \frac{y - x}{|y - x|} - \frac{x - b_x}{d(x, \partial\Omega)}.$$

We want to show that the limsup

$$\limsup_{\epsilon \to 0} \mathcal{H}^{n-1}(U \cap g_y^{-1}(\epsilon))$$

is uniformly bounded by $\mathcal{H}^{n-1}(\overline{U} \cap g_y^{-1}(0))$, as we know from e.g. [1] that this last set is contained in the boundary of a convex set, which is contained in a ball $B(y, 4 \cdot 2^k)$. By monotonicity of perimeter of nested convex sets, we get that this last Hausdorff measure is at most $\lesssim 2^{k(n-1)}$, which then provides good bounds for the limsup.

2. Theorem

Let D be an open domain, $g : D \to \mathbb{R}$ Lipschitz continuous in D. Let $U \subset\subset D$ be a bounded, Lipschitz domain such that $g|_{\tilde{U}}$ is C^1 with Lipschitz gradient on an open set $U \subset\subset \tilde{U}$. Suppose also that $|\nabla g| \sim C$ in $\tilde{U}$. Then it holds that

$$\limsup_{\epsilon \to 0} \mathcal{H}^{n-1}(U \cap g^{-1}(\epsilon)) \leq \mathcal{H}^{n-1}(\overline{U} \cap g^{-1}(0)).$$

Proof. As g is of class C^1 on a larger open set $\tilde{U}$ and the gradient of g does not vanish on $\tilde{U}$, the level sets $g^{-1}(\epsilon) \cap \tilde{U}$ are all C^1-smooth hypersurfaces in $\tilde{U}$. Consider then the thickening $V_\delta(g) = \{x \in \tilde{U} \colon d(x, g^{-1}(0)) < \delta\}$.

Claim 1. *There is $\epsilon > 0$ such that, for $\epsilon < \epsilon_0$, $g^{-1}(\epsilon) \cap U \subset V_\delta(g)$.*

Proof of Claim 1. If not, then there is a sequence $\epsilon_k \to 0$ and $x_k \in g^{-1}(\epsilon_k) \cap U$ with $d(x_k, g^{-1}(0)) > \delta$. But then, as U is bounded, we may pass to a subsequence to suppose that $x_k \to \tilde{x} \in \overline{U}$. We obviously have $\tilde{x} \in g^{-1}(0) \cap \overline{U}$, which contradicts the fact that $d(\tilde{x}, g^{-1}(0)) = \lim d(x_k, g^{-1}(0)) \geq \delta$. □

We now notice that the sets $g^{-1}(\epsilon) \cap \overline{U}$ are compact, and as $|\nabla g| \sim C$, localizing into charts shows that they are all sets of finite perimeter. In order to conclude the assertion, we need a couple more properties.

Firstly, we have that the sets $g^{-1}(\epsilon) \cap U$ have at most $c(U)$ connected components, independently of ϵ. This follows, for instance, from the fact that these sets are all (restrictions of) a differentiable family of C^1 hypersurfaces to a compact set, and the local form of submersions. This fact alone plainly implies, together with compacity, that there is a constant $d(U) > 0$ such that the distance between two such components is always greater than $d(U) > 0$ if ϵ is small.

Define $\tilde{U}_N = \{z \in \tilde{U} \colon d(z, \partial U) > \frac{1}{N}\}$. We first notice that $\tilde{U}_N \supset U$ for N large. Now note that, if δ is small enough, the set $V_\delta^N(g) := V_\delta(g) \cap U_N$ can be partitioned into $V_\delta^{N,+} \cup V_\delta^{N,-} \cup (g^{-1}(0) \cap \tilde{U}_N)$, where $(g^{-1}(0) \cap \tilde{U}_N) \in \partial V_\delta^{N,\pm}$. This follows from the fact that $g^{-1}(0) \cap \tilde{U}$ is a compact, orientable hypersurface for each connected component, so that there is a finite number of balls $B_i, i = 1, \cdots, N$ such that the "upper" and "lower" parts $B_i^{\pm}$ are well defined, and gluing them together works by orientability. Furthermore, it is easy to see that each $V_\delta^{N,\pm}(g)$ is a C^1 domain, except for a set of Hausdorff dimension $n-2$, which implies that both are domains so that the almost-C^1 version of Green's theorem holds.

The considerations above imply that, within a connected component of $V_\delta^N(g)$, $\delta \ll d(U)$ small, there is *at most* one connected component of $g^{-1}(\epsilon) \cap \tilde{U}_N$, where ϵ is also very small.

Claim 2. *There is $\epsilon_1 > 0$ such that either $g^{-1}(\epsilon) \cap \tilde{U}_N \subset V_\delta^{N,+}$, $\forall \epsilon < \epsilon_1$ or $g^{-1}(\epsilon) \cap \tilde{U}_N \subset V_\delta^{N,-}$, $\forall \epsilon < \epsilon_1$.*

Proof of Claim 2. First, notice that by connectivity, each set $g^{-1}(\epsilon) \cap \tilde{U}_N$ belongs to *exactly one* side $V_\delta^{N,\pm}$. Define the the sets $I_\pm = \{\alpha \in \mathbb{R}\backslash\{0\} \colon \alpha \text{ small}, g^{-1}(\alpha) \cap \tilde{U}_N \subset V_\delta^{N,\pm}\}$. Clearly, they are both open. They are also closed, as the sets that do not belong to either would have to be contained in $g^{-1}(0)$. It means that one of the $I_\pm$ equals $(0, \epsilon_0)$ and the other $(-\epsilon_0, 0)$. This implies the desired assertion. □

We are now able to finish. By Claim 1, let ϵ_2 be so that $g^{-1}(\epsilon) \cap \tilde{U}_N \subset V_{\delta^2}^N$ if $\epsilon < \epsilon_2$. By Claim 2, suppose without loss of generality that $g^{-1}(\epsilon) \cap \tilde{U}_N \subset V_\delta^{N,+}$ for $\epsilon < \epsilon_2$. We assume also without loss of generality that the connectivity constant above is $c(U) = 1$, as the general case follows by repeating the following argument on each connected component.

Let $\phi : \tilde{U}_N \to \mathbb{R}$ be a smooth function so that $\phi \equiv 1$ in $V_{\delta^2}^{N,+}$ and $\phi \equiv 0$ in $\tilde{U}_N \backslash V_{2\delta^2}^{N,+}$. Define the *normal field* associated to g at every point as $\nu_g(x) = \frac{\nabla g(x)}{\|\nabla g(x)\|}$. Define $u = \phi \cdot \nu_g$. By the definition of ϕ, we have:

$$\mathcal{H}^{n-1}(\partial V_{2\delta^2}^{N,+} \cap \partial \tilde{U}_N) + \mathcal{H}^{n-1}(g^{-1}(0) \cap \tilde{U}_N) \geq \int_{\partial V_\delta^{N,+}} \langle u, \nu_{V_{\delta^2}^{N,+}} \rangle \cdot d\mathcal{H}^{n-1}.$$

On the other hand, the integral above equals, by the almost-C^1 version of the divergence theorem and dominated convergence,

$$\int_{V_\delta^{N,+}} \operatorname{div}(u)\, dx = \lim_{\epsilon \to 0} \int_{V_\epsilon} \operatorname{div}(u)\, dx,$$

where V_ϵ is the part of $V_\delta^{N,+}$ "above" $g^{-1}(\epsilon) \cap \tilde{U}_N$. Once again by divergence,

$$\int_{V_\epsilon} \operatorname{div}(u)\, dx = \int_{\partial V_\epsilon} \langle u, \nu_{V_\epsilon} \rangle\, d\mathcal{H}^{n-1} \geq \mathcal{H}^{n-1}(g^{-1}(\epsilon) \cap U).$$

Putting together implies

$$\limsup_{\epsilon \to 0} \mathcal{H}^{n-1}(g^{-1}(\epsilon) \cap U) \leq \mathcal{H}^{n-1}(\partial V_{2\delta^2}^{N,+} \cap \partial \tilde{U}_N) + \mathcal{H}^{n-1}(g^{-1}(0) \cap \tilde{U}_N).$$

Notice that the same applies to any domain W such that $U \subset\subset W \subset\subset \tilde{U}_N$. Therefore, by taking W so that $g^{-1}(0)$ intersects ∂W (at most) transversally implies that

$$\limsup_{\delta \to 0} \mathcal{H}^{n-1}(V_{\delta^2}^{N,+} \cap \partial W) = 0.$$

Therefore, by finite perimeter, taking $N \to 0$ and if we choose $\delta > 0$ so that the $(n-1)-$Hausdorff measure of the sets above is less than $\eta \cdot \mathcal{H}^{n-1}(g^{-1}(0) \cap \overline{U})$, we get

$$\limsup_{\epsilon \to 0} \mathcal{H}^{n-1}(g^{-1}(\epsilon) \cap U) \leq (1+\eta)\mathcal{H}^{n-1}(g^{-1}(0) \cap \overline{U}).$$

By making $\eta \to 0$ one obtains the result. $\square$

References

[1] J.P.G. Ramos, O. Saari and J. Weigt, *Smoothing of singular functions through local fractional maximal operator.* Preprint.