Derived Categories

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Remerciements

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Introduction

Derived categories were introduced in the early sixties by A. Grothendieck. His goal was to extend Serre's duality theorem to obtain a global theory of duality in cohomology for algebraic varieties admitting arbitrary singularities ([6]). Grothendieck's student J.L. Verdier developed the basic theory first during his Ph.D. thesis. His works appeared in SGA 4½ in 1977 ([15]), and later published in Astérisque in 1996 ([16]). Verdier introduced the new concepts of *triangulated categories* and *derived categories of abelian categories*, and their uses in the proof of Grothendieck's duality theorem are exposed in Hartshorne's *Residues and duality* ([7]), lecture notes on a seminar that took place in Havard in 1963-64.

Since then, theory of derived categories have become wide-spread in algebraic geometry, and recently have found success in areas nearer to physics ([10]). Considered as the right framework for any kind of derived functors, the study of derived categories of coherent sheaves on (projective) varieties, as done in this text, originally goes back to Mukai in the eighties. He constructed geometrically motivated equivalences between derived categories of non-isomorphic varieties ([12], [13]). In 1997, Bondal and Orlov showed that the derived category of coherent sheaves turns out to be a complete invariant for projective varieties whenever the canonical bundle is either ample or anti-ample ([1]). The goal of this text is to study this last result.

This text will be splitted in three chapters. Both chapters I and III are mainly based on Huybrechts's *Fourier-Mukai Transforms in Algebraic Geometry* ([9], chapters 1-4), and chapter II is based on Hartshorne's *Algebraic Geometry* ([8]) and Görtz and Wedhorn's *Algebraic Geometry I: Schemes* ([4]). We refer to Bourbaki ([2]) and Matsumura ([11]) for general commutative algebra results, and to Gelfand and Manin ([3]) for homological algebra results. Finally, we refer to Hartshorne's *Residues and duality* ([7]) for more details on derived categories and localization theory.

Chapter I is devoted to general theory of triangulated and derived categories. Given an abelian category $\mathcal{A}$, we construct the associated derived category $\mathrm{D}(\mathcal{A})$, which objects are complexes of objects in $\mathcal{A}$ and arrows are homotopy classes of morphisms of complexes to which we added "inverses" of quasi-isomorphisms. This new category admits a structure of triangulated category, and has many properties that we briefly study. Finally, given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we show how to lift it to a "derived" functor $RF : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ between the associated derived category, and we present the Grothendieck's spectral sequence which is a fundamental tool in this theory.

In Chapter II, we study the fundation of algebraic geometry. We briefly recall the notion of varieties seen as locally ringed spaces and we introduce the language of schemes. Given a (noetherian) scheme $(X, \mathcal{O}_X)$, we define the notion of (quasi)-coherent $\mathcal{O}_X$-modules, and point out properties of them. In the last part, we consider closed immersions $X \hookrightarrow \mathbb{P}^n$ and give conditions, when k is algebraically closed, to obtain such embeddings. Finally, we state the well-know Serre duality theorem: when X is a smooth projective variety of dimension n, for any interger k and any locally free sheaf $\mathcal{F}$ we obtain an isomorphism

$$H^k(X, \mathcal{F}) \xrightarrow{\sim} H^{n-k}(X, \mathcal{F}^\vee \otimes \omega_X).$$

In Chapter III, we focus our attention to the bounded derived category $\mathrm{D}^b(X) = \mathrm{D}^b(\mathbf{Coh}(X))$ of the abelian category of coherent sheaves on a smooth projective variety X. We first expose properties of this category, and show how to define Grothendieck's spectral sequences and Serre duality in this context. The category $\mathbf{Coh}(X)$ has not enough injectives neither projectives in general, but we still can construct derived functors ($R\operatorname{Hom}(\ ,\ ), R\mathcal{H}om(\ ,\ ), \otimes^L$) that are needed in the sequel. We finish this text by presenting results due to Bondal and Orlov ([1]). We introduce the (intrinsic) notion of point like and inversible objects in $\mathrm{D}^b(X)$ and prove the main theorem: if X and Y are smooth projective variety, if ω_X or ω_X^* is ample, then any exact equivalence $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ yields an isomorphism of varieties $X \simeq Y$.

Part I
Derived categories

In this part, we will introduce the notion of derived category of an arbitrary abelian category, and study its structure of triangulated category. We will also define derived functors and introduce spectral sequences, a relevant tool that we will use in the rest of the text. A deeper study of this theories can be found in [3] and [7].

1 Triangulated and derived categories

1.1 Additive and abelian categories

All the categories we will consider are supposed *locally small*, *i.e.* for any two objects A, B the collection $\mathrm{Hom}(A, B)$ is a set.

Definition 1.1. A category $\mathcal{C}$ is an *additive category* if for every two objects $A, B \in \mathcal{C}$ the set $\mathrm{Hom}(A, B)$ is endowed with a structure of abelian group and the following three conditions are satisfied:

1. The composition $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is bilinear, *i.e.* we have $(f + g) \circ h = f \circ h + g \circ h$ and $f \circ (h + l) = f \circ h + f \circ l$.

2. There exist an object $0 \in \mathcal{C}$ which is both *inital* and *terminal*, *i.e.* for all objects $A \in \mathcal{C}$, we have $0 = \mathrm{Hom}(A, 0) \simeq \mathrm{Hom}(0, A)$.

3. For any two objects $A_1, A_2 \in \mathcal{C}$, there exist an object B, called the *biproduct* of A_1 and A_2, and arrows $j_i : A_i \to B$ and $p_i : B \to A_i$, $i = 1, 2$, verifying the following properties:

   - For every object $D \in \mathcal{C}$ and arrows $l_i : A_i \to D$, there exist a unique arrow $l : B \to D$ such that $l_i = l \circ j_i$.
   - For every object $D \in \mathcal{C}$ and arrows $q_i : D \to A_i$, there exist a unique arrow $q : D \to B$ such that $q_i = p_i \circ q$.

   Such an object is unique and is denoted $A_1 \oplus A_2$.

Remark 1.2. Given a field k, one can define similarly a *k-linear* category asking the Hom sets to be k-vector spaces and the composition to be k-bilinear.

Definition 1.3. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories (resp. k-linear categories) is said to be *additive* if the induced maps $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ are group homomorphisms (resp. k-linear maps).

Definition 1.4. Let $\mathcal{C}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathcal{C} \to \mathcal{C}$ such that for any two objects $A, B \in \mathcal{C}$ there exists an isomorphism of k-vector spaces

$$\eta_{A,B} : \mathrm{Hom}(A, B) \xrightarrow{\simeq} \mathrm{Hom}(B, S(A))^*$$

which is functorial in A and B, where the * denotes the dual vector space.

To avoid any troubles with the dual, we will usually assume that all Hom's are finite dimensional.

Proposition 1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two k-linear categories over a field k with finite-dimensional* Hom*'s. If $\mathcal{C}$ and $\mathcal{D}$ are endowed with Serre functors $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ respectively, then any k-linear equivalence*

$$F : \mathcal{C} \to \mathcal{D}$$

commutes with the Serre functors, i.e. there exists an isomorphism

$$F \circ S_{\mathcal{C}} \simeq S_{\mathcal{D}} \circ F.$$

Proof. Since F is fully faithful, for two objects $A, B \in \mathcal{C}$ we have

$$\mathrm{Hom}(A, S(B)) \simeq \mathrm{Hom}(F(A), F(S(B))) \text{ and } \mathrm{Hom}(B, A) \simeq \mathrm{Hom}(F(B), F(A)).$$

On the other hand we have

$$\mathrm{Hom}(A, S(B)) \simeq \mathrm{Hom}(B, A)^* \text{ and } \mathrm{Hom}(F(B), F(A)) \simeq \mathrm{Hom}(F(A), S(F(B)))^*.$$

Thus we have a functorial (in A and B) isomorphism

$$\mathrm{Hom}(F(A), F(S(B))) \simeq \mathrm{Hom}(F(A), S(F(B))).$$

Since F is essentially surjective, any object in $\mathcal{D}$ is isomorphic to an object of the form $F(A)$ for some $A \in \mathcal{C}$. Thus, applying the Yoneda lemma to the last isomorphism, we obtain that $F \circ S \simeq S \circ F$.

□

Definition 1.6. An additive category $\mathcal{A}$ is called *abelian* if every arrow $f : A \to B$ in $\mathcal{A}$ admits a kernel and a cokernel and the natural arrow $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Thus, for any arrow $f : A \to B$ we have the following diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] & A \arrow[rr, "f"] \arrow[dr] & & B \arrow[r, "\pi"] & \mathrm{coker}(f). \\
& & \mathrm{coker}(i) \arrow[r, "\simeq"] & \ker(\pi) \arrow[u] &
\end{tikzcd}$$

A strong motivation in the study of abelian categories lies in the following definition:

Definition 1.7. Let $\mathcal{A}$ be an abelian category. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$ is called *exact in* B if $\ker g = \mathrm{Im}\, f$.

A sequence

$$\cdots \to A^{n-1} \to A^n \to A^{n+1} \to \cdots$$

is called *exact* if it is exact in A^i for all $i \in \mathbb{Z}$. Relevant particular cases are *short exact sequences* which are exact sequences of the shape:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0.$$

In this case, f is *injective* (*i.e.* $\ker(f) = 0$) and h is *surjective* (*i.e.* $\mathrm{coker}\, h = 0$).

Definition 1.8. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *left exact* (resp. *right exact*) if any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0$$

is sent to an exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(h)} F(C)$$

(resp. to an exact sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(h)} F(C) \longrightarrow 0).$$

Example 1.9. Let A be an object in a abelian category $\mathcal{A}$. Then the functors $\mathrm{Hom}(A, \ \ )$ and $\mathrm{Hom}(\ \ , A)$, which take values in the category of abelian groups **Ab**, are left exact, but not right exact in general.

1.2 Triangulated categories

Now we will talk about the structure of triangulated category: it is in some sense a generalization of the notion of exact sequences extended to general additive categories.

Definition 1.10. Let $\mathcal{D}$ be an additive category. The structure of *triangulated category* on $\mathcal{D}$ is given by an additive equivalence $T : \mathcal{D} \to \mathcal{D}$, called the *shift functor*, and a set of diagrams of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A),$$

called *distinguished triangles*, where A, B, C are objects of $\mathcal{D}$. A morphism between two triangles is given by a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & T(A'),
\end{tikzcd}$$

and it is an isomorphism if the vertical arrows are isomorphisms. We ask the distinguished triangles to satisfy the axioms TR-1 to TR-4 below :

TR-1 :

- Any triangle for the form

  $$A \xrightarrow{\mathrm{Id}} A \longrightarrow 0 \longrightarrow T(A)$$

  is distinguished.
- Any triangle isomorphic to a distinguished triangle is distinguished.
- Any arrow $f : A \to B$ can be completed to a distinguished triangle

  $$A \xrightarrow{f} B \longrightarrow C \longrightarrow T(A).$$

TR-2 : The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{-T(f)} T(B)$$

is distinguished.

TR-3 : Suppose there exists a commutative diagram of distinguished triangles with vertical arrow f and g:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dotted, "h"'] & T(A) \arrow[d, "T(f)"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & T(A'). \end{tikzcd}$$

Then there exists an (non-unique) arrow $h : C \to C'$ such that the diagram commutes (in particular, we obtain a morphism of distinguished triangles).

TR-4 : We will omit this axiom because it is the most complicated to state and we will not use it in this text. It is called the *octahedron* axiom. The idea is the following: assume that we are in an abelian category, and replace the notion of distinguished triangles by the notion of exact sequences. Given inclusions $A \subseteq B \subseteq C$, TR-4 ask that if the sequences $A \to B \to B/A$, $B \to C \to C/B$ and $A \to C \to C/A$ are exact, so is $B/A \to C/A \to C/B$.

For now on, let's write $A[1]$ for $T(A)$ and $f[1]$ for $T(f)$. More generally, for any integer k we will denote $A[k]$ for $T^k(A)$ and $f[k]$ for $T^k(f)$, where T^k is $T^{\circ k}$ when k is nonnegative and $(T^{-1})^{\circ -k}$ when k is negative.

Remark 1.11.
- First, note that since T is an equivalence, any object A in T is isomorphic to the object $(A[-1])[1]$ (*i.e.* an object in the image of T) and is also isomorphic to the object $(A[1])[-1]$, thus, using the axiom that any triangle isomorphic to a distinguished triangle is distinguished, we can extend the axiom TR-2 to the following: a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

is distinguished if and only if any triangle extracted from the sequence

$$\cdots \longrightarrow B[-1] \xrightarrow{-g[-1]} C[-1] \xrightarrow{-h[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1] \longrightarrow \cdots$$

is distinguished.

- In the same vein, one can prove from the axiom TR-1 that the triangles

$$A \xrightarrow{-\mathrm{Id}} A \longrightarrow 0 \longrightarrow T(A)$$

and

$$0 \longrightarrow A \xrightarrow{\pm\mathrm{Id}} A \longrightarrow 0$$

are also distinguished.

- The axiom TR-3 can be generalized in the following sense: given a commutative diagram of distinguished triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & T(A'), \end{tikzcd}$$

if two of the vertical arrows f, g, h exist so does the third.

Proposition 1.12. *Let $A \to B \to C \to A[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then for any object $A_0 \in \mathcal{D}$ the following induced sequences are exact sequences of abelian groups:*

$$\mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C)$$

$$\mathrm{Hom}(C, A_0) \longrightarrow \mathrm{Hom}(B, A_0) \longrightarrow \mathrm{Hom}(A, A_0).$$

Proof. First notice a general fact: for any distinguished triangle as the one in the proposition, the composition $A \to B \to C$ is 0. It suffices to apply TR-1 to the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, dotted] & A[1] \arrow[d] \\ A \arrow[r, "f"'] & B \arrow[r] & C \arrow[r] & A[1]. \end{tikzcd}$$

Now, assume $f : A_0 \to B$ composed with $B \to C$ is $0 : A_0 \to C$. Then apply TR-1 and TR-3 to the diagram:

$$\begin{tikzcd} A_0 \arrow[r, "\mathrm{Id}"] \arrow[d, dotted] & A_0 \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ A \arrow[r] & B \arrow[r] & C, \end{tikzcd}$$

we obtain a lift of f to an arrow $A_0 \to A$. The second assertion can be proved in a similar way. $\square$

Remark 1.13. Once again, applying TR-2 one can show that the sequence $\mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, C) \to \mathrm{Hom}(A_0, A[1])$ is also exact in **Ab**. In particular, we actually obtain *long* exact sequences of abelian groups.

There is a lot of properties about distinguished triangles we could deduce from the last proposition, but the following ones are the most important:

Lemma 1.14. *Let $A \to B \to C \to A[1]$ be a distinguished triangle.*

1. *$A \to B$ is an isomorphism if and only if $C \simeq 0$.*
2. *If $C \to A[1]$ is trivial, then the triangle splits, i.e. is given by a decomposition $B \simeq A \oplus C$.*
3. *Consider a morphism of distinguished triangles*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & A[1] \arrow[d, "f[1]"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

If two of the vertical arrows f, g, h are isomorphisms then so is the third.

Proof. 1. Consider the following sequence:

$$\mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C) \longrightarrow \mathrm{Hom}(A_0, A[1]) \longrightarrow \mathrm{Hom}(A_0, B[1]).$$

Since the morphisms are functorial, the result follows from the Yoneda lemma.

2. Applying Proposition 1.12, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C) \longrightarrow 0.$$

Since this sequence splits, we have a functorial isomorphism

$$\mathrm{Hom}(A_0, B) \simeq \mathrm{Hom}(A_0, A) \oplus \mathrm{Hom}(A_0, C) = \mathrm{Hom}(A_0, A \oplus C),$$

so the result follows from the Yoneda lemma again.

3. This can be proved applying $\mathrm{Hom}(A_0, \ )$ on the diagram and using the five lemma. $\square$

Definition 1.15. An additive functor $F : \mathcal{D} \to \mathcal{D}'$ between triangulated categories is called *exact* if the following conditions are satisfied:

- There exist a natural isomorphism $F \circ T_{\mathcal{D}} \simeq T_{\mathcal{D}'} \circ F$.
- Any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ is mapped to a distinguished triangle $F(A) \to F(B) \to F(C) \to F(A)[1] \simeq F(A[1])$ in $\mathcal{D}'$.

Proposition 1.16. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If $F \dashv H$, then $H : \mathcal{D}' \to \mathcal{D}$ is exact.*

Proof. First let's prove the commutativity between H and the shift functors T and T' of $\mathcal{D}$ and $\mathcal{D}'$ respectively. We have the functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}(A, H(T'(B))) &\simeq \mathrm{Hom}(F(A), T'(B)), \\ &\simeq \mathrm{Hom}(T'^{-1}(F(A)), B), \\ &\simeq \mathrm{Hom}(F(T^{-1}(A)), B), \\ &\simeq \mathrm{Hom}(T^{-1}(A), H(B)), \\ &\simeq \mathrm{Hom}(A, T(H(B))). \end{aligned}$$

By the Yoneda lemma, we get the isomorphism

$$H \circ T' \xrightarrow{\simeq} T \circ H.$$

Now, let $A \to B \to C \to A[1]$ be a distinguished triangle in $\mathcal{D}$. The arrow $H(A) \to H(B)$ can be completed in $\mathcal{D}'$ in a distinguished triangle

$$H(A) \longrightarrow H(B) \longrightarrow C_0 \longrightarrow H(A)[1].$$

Using the adjunction morphisms $F(H(A_0)) \to A_0$ for any $A_0 \in \mathcal{D}$ we get the commutative diagram

$$\begin{tikzcd} F(H(A)) \arrow[r] \arrow[d] & F(H(B)) \arrow[r] \arrow[d] & F(H(C)) \arrow[r] \arrow[d, dotted, "\xi"'] & F(H(A))[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1], \end{tikzcd}$$

which can be completed since both triangles are distinguished. Applying H and using the adjunction $h : \mathrm{Id} \to H \circ F$ we get the diagram:

$$\begin{tikzcd} H(A) \arrow[r] \arrow[d] & H(B) \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "h_{C_0}"'] & H(A)[1] \arrow[d] \\ HFHA \arrow[r] \arrow[d] & HFHB \arrow[r] \arrow[d] & HFC_0 \arrow[r] \arrow[d, "H(\xi)"'] & HFHA[1] \arrow[d] \\ H(A) \arrow[r] & H(B) \arrow[r] & H(C) \arrow[r] & H(A)[1]. \end{tikzcd} \tag{1}$$

Since the last row is not distinguished, we can not conclude using Lemma 1.14. But we know that for any $A_0 \in \mathcal{D}$ the sequence

$$\mathrm{Hom}(F(A_0), B) \to \mathrm{Hom}(F(A_0), C) \to \mathrm{Hom}(F(A_0), A[1])$$

is exact, so using the adjunctions we obtain that the sequence

$$\mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, B[1])$$

is exact, and thus we can apply the five lemma on the diagram obtain by applying $\mathrm{Hom}(A_0, \ )$ to the diagram (1). We get $\mathrm{Hom}(A_0, C_0) \simeq \mathrm{Hom}(A_0, H(C))$ for all A_0 and thus

$$H(\xi) \circ h_{C_0} : C_0 \xrightarrow{\simeq} H(C) .$$

This achieves the proof because $H(A) \to H(B) \to H(C) \to H(A)[1]$ is distinguished since it is isomorphic to a distinguished triangle. □

Definition 1.17. A subcategory $\mathcal{D}' \subseteq \mathcal{D}$ of a triangulated category is a triangulated subcategory if $\mathcal{D}'$ admits a structure of triangulated category such that the inclusion is exact.

Proposition 1.18. *If $\mathcal{D}' \subseteq \mathcal{D}$ is a full subcategory of a triangulated category $\mathcal{D}$, then it is a triangulated subcategory if and only if it is invariant by the shift functor and for any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$, the object C is isomorphic to an object in $\mathcal{D}'$.*

Proof. If $\mathcal{D}'$ is a triangulated subcategory, then it is invariant by the shift functor, and considering a distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$, the arrow $A \to B$ can be completed to a distinguished triangle $A \to B \to C_0 \to A[1]$ in $\mathcal{D}'$ which is also distinguished in $\mathcal{D}$ since the inclusion is exact. Thus we get the commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted] & A[1] \arrow[d] \\
A \arrow[r] & B \arrow[r] & C_0 \arrow[r] & A[1].
\end{tikzcd}$$

Using the axiom TR-3 one can complete the diagram to a morphism of distinguished triangle, and since all vertical arrows are isomorphism, so is $C \to C_0$. Conversely, the second hypothesis tells exactly that the third TR-1 axiom hold, and all other axioms follow from the fact that $\mathcal{D}'$ is full and invariant under the shift functor. □

Definition 1.19. A collection Ω of objects in a triangulated category $\mathcal{D}$ is a *spanning class* of $\mathcal{D}$ (or *spans* $\mathcal{D}$) if for all $B \in \mathcal{D}$ the following two conditions holds:

1. If $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

2. If $\mathrm{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

Note that these two conditions are equivalent if $\mathcal{D}$ is equipped with a Serre functor.

1.3 Derived categories

We consider an abelian category $\mathcal{A}$.

Definition 1.20.

- A *(differential) complex* $A^\bullet$ in $\mathcal{A}$ is the data of a family $(A^n)_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$ and a family $(d^n : A^n \to A^{n+1})_{n\in\mathbb{Z}}$ of arrows verifiying $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

- A morphism $f : A^\bullet \to B^\bullet$ between two complexes $A^\bullet = ((A^n), d_A^n)_{n\in\mathbb{Z}}$ and $B^\bullet = (B^n, d_B^n)_{n\in\mathbb{Z}}$ is given by a family $(f^n : A^n \to B^n)_{n\in\mathbb{Z}}$ such that for any $n \in \mathbb{Z}$ the diagram

$$\begin{tikzcd}
A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B^n \arrow[r, "d_B^n"] & B^{n+1}
\end{tikzcd}$$

commutes.

- We denoted by $\mathrm{Kom}(\mathcal{A})$ the category of complexes of $\mathcal{A}$ whose objects are complexes and whose arrows are morphisms of complexes.

For any $n \in \mathbb{Z}$, the object object A^n is called *object in degree* n. The arrows d^n are sometimes called *the differentials* of the complex. We will often define a complex $A^\bullet$ as a diagram

$$\cdots \to A^{n-1} \to A^n \to A^{n+1} \to \cdots$$

and we will omit the index of the arrows d when the context is clear.

Proposition 1.21. *The category of complexes* $\mathrm{Kom}(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *is abelian.*

Proof. The proof is straightforward: all the structure of an abelian category on $\mathrm{Kom}(\mathcal{A})$ is given on $\mathcal{A}$ degree by degree. For instance, $\ker f = (\ker f^n)_{n\in\mathbb{Z}}$. □

Remark 1.22. The category $\mathcal{A}$ embeds into $\mathrm{Kom}(\mathcal{A})$ as a full subcategory by identifying an object $A \in \mathcal{A}$ with the complex

$$\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots$$

concentrated in degree 0 with trivial differentials.

Definition 1.23. Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ be a complex. Then we define the complex $A^\bullet[k]$ for any $k \in \mathbb{Z}$ by $A^\bullet[k]^n := A^{k+n}$ and $d_{A[k]} = (-1)^k d_A^{i+k}$.

Proposition 1.24. *The shift functor* $A^\bullet \mapsto A^\bullet[1]$ *is an equivalence of abelian categories.*

Proof. The inverse functor is given by $A^\bullet \mapsto A^\bullet[-1]$, details are left to the reader. □

Definition 1.25. Let $A^\bullet$ be a complex in $\mathrm{Kom}(\mathcal{A})$. We define the n^{th} *cohomology object* $H^n(A^\bullet)$ as the quotient

$$H^n(A^\bullet) := \frac{\ker(d^n)}{\mathrm{Im}(d^{n-1})} \in \mathcal{A}.$$

To be precise, a more category-theoretic definition would be

$$H^n := \mathrm{coker}(\mathrm{Im}(d^{n-1}) \hookrightarrow \ker(d^n))$$

Proposition 1.26. *For all* $n \in \mathbb{Z}$, *a morphism of complexes* $f : A^\bullet \to B^\bullet$ *induces a morphism in cohomology*

$$H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet).$$

Proof. This is straightforward using the fact that any morphism of complexes commutes with the differentials. □

Proposition 1.27. *Any short exact sequence*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

induces a long exact sequence

$$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \cdots .$$

This is a classical result of homological algebra using diagram chasing. It is a consequence of the famous *snake lemma*. See ([3], §5, ex.7).

Definition 1.28. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* (or *qis* for short) if for all $n \in \mathbb{Z}$ the induced arrow $H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$ is an isomorphism.

In order to study cohomology, we would like quasi-isomorphisms to be actual isomorphisms. To do so, we will construct a new category, called the *derived category* of the abelian category $\mathcal{A}$, in which all quasi-isomorphisms are invertible. This will be done in several steps.

Definition 1.29.

- Two morphisms of complexes

$$f, g : A^\bullet \to B^\bullet$$

are called *homotopically equivalent*, denoted $f \sim g$, if there exists a collection of homomorphisms $h^n : A^n \to B^{n-1}$ for all $n \in \mathbb{Z}$ such that

$$f^n - g^n = h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n.$$

Such a family $(h^n)_{n\in\mathbb{Z}}$ is called a *homotopy* between f and g.

- The *homotopy category* of complexes $\mathrm{K}(\mathcal{A})$ is the category whose objects are the objects of $\mathrm{Kom}(\mathcal{A})$ and for all $A^\bullet, B^\bullet \in \mathrm{K}(\mathcal{A})$ we have

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim .$$

It can be verified that $\mathrm{K}(\mathcal{A})$ is indeed a category, in particular that homotopy equivalence is indeed an equivalence relation. A big part of the interest in homotopy of complexes lies in the following:

Proposition 1.30. *Let $f, g : A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$. If $f \sim g$ then $H^n(f) = H^n(g)$ for all $n \in \mathbb{Z}$.*

Proof. By definition of homotopy, $f^n - g^n = h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n$ for some homotopy $(h^n)_n$ then $f^n - g^n$ sends $\ker(d_A^n)$ into $\mathrm{Im}(d_B^{n-1})$, thus $H^n(f - g) = 0$. □

Corollary 1.31. *If $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ verify $f \circ g \sim \mathrm{Id}_B$ and $g \circ f \sim \mathrm{Id}_A$ then f and g are quasi-isomorphisms and $H^n(f)^{-1} = H^n(g)$.*

Remark 1.32. If $\mathcal{A}$ is a general additive category, we still can consider the category of complexes $\mathrm{Kom}(\mathcal{A})$ and the definition of $\mathrm{K}(\mathcal{A})$ also makes sense.

Definition 1.33. Let $\mathcal{A}$ be an abelian category. Then we define the *derived category* of $\mathcal{A}$, denoted $\mathrm{D}(\mathcal{A})$ to be the category whose objects are the ones of $\mathrm{Kom}(\mathcal{A})$, *i.e.* :

$$\mathrm{Ob}(\mathrm{D}(\mathcal{A})) = \mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A})),$$

and arrows are defined as follows. Let $A^\bullet, B^\bullet$ be two objects in $\mathrm{D}(\mathcal{A})$. The set of morphism $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ is defined as the set of equivalent classes of diagrams (called *roofs*) of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

where $C^\bullet$ is another object in $\mathrm{Kom}(\mathcal{A})$, s is a quasi-isomorphism and f is a morphism. Two such diagrams are equivalent if they are dominated in $\mathrm{K}(\mathcal{A})$ by a third one of the same sort, *i.e.* if there exists a commutative diagram in $\mathrm{K}(\mathcal{A})$ of the form

$$\begin{tikzcd} & D \arrow[dl, "u"'] \arrow[dr] & \\ C^\bullet \arrow[d, "s"'] \arrow[drr] & & C'^\bullet \arrow[d] \arrow[dll] \\ A^\bullet & & B^\bullet \end{tikzcd}$$

such that $s \circ u$ is a quasi-isomorphisms.

The composition of two morphisms

$$\begin{tikzcd}[column sep=small] & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

is given by a commutative diagram in $\mathrm{K}(\mathcal{A})$ of the form

$$\begin{tikzcd}[column sep=small] & & C_0^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dotted, bend right, "qis"'] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet. \end{tikzcd}$$

Our goal now is to check that these definitions really define a category, in particular that the composition exists and is unique up to equivalence. To do so, we need to introduce the mapping cone which plays a central role in the definition of triangulated structures on $\mathrm{K}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})$.

Definition 1.34. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Its *mapping cone* is the complex $\mathrm{C}(f)$ defined by

$$\mathrm{C}(f)^n = A^{n+1} \oplus B^n \quad \text{and} \quad d^n_{\mathrm{C}(f)} := \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

Remark 1.35.
- The mapping cone $\mathrm{C}(f)$ is indeed a complex: $d_B^{n+1} \circ f^{n+1} = f^{n+2} \circ d_A^{n+1}$ since f is a morphism of complexes.
- We have natural morphisms of complexes

  $$\tau : B^\bullet \to \mathrm{C}(f) \quad \text{and} \quad \pi : \mathrm{C}(f) \to A^\bullet[1]$$

  given by the natural injection $B^n \to A^{n+1} \oplus B^n$ and the natural projection $A^{n+1} \oplus B^n \to A^{n+1}$.
- The composition $A^\bullet \to B^\bullet \to \mathrm{C}(f)$ is nullhomotopic (*i.e.* homotopic to the trivial map), such an homotopy is given by $(\iota_n : A^n \to A^n \oplus B^{n-1})_{n \in \mathbb{Z}}$. Indeed, we have :

  $$\begin{tikzcd} \cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r, "d"] \arrow[dl, "\iota_n"'] & A^{n+1} \arrow[r] \arrow[dl, "\iota_{n+1}"'] & \cdots \\ \cdots \arrow[r] & A^n \oplus B^{n-1} \arrow[r, "d'"'] & A^{n+1} \oplus B^n \arrow[r] & A^{n+2} \oplus B^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

  and we have $\iota_{n+1} \circ d = (d_A^{n+1}, 0)$ and $d' \circ \iota_n = (-d_A^{n+1}, f^n)$.
- The sequence $0 \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1] \to 0$ is exact: it comes from the fact that the composition $M \to M \oplus N \to N$ is 0 in any additive category. In particular, we have a long exact sequence

  $$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(\mathrm{C}(f)) \to H^{n+1}(A^\bullet) \to \cdots .$$

- Using the previous long exact sequence, we obtain that f is a quasi-isomorphism if and only if $H^n(\mathrm{C}(f)) = 0$ for all $n \in \mathbb{Z}$.

- By construction, any commutative diagram

$$\begin{tikzcd} A_1^\bullet \arrow[r, "f_1"] \arrow[d] & B_1^\bullet \arrow[r] \arrow[d] & \mathrm{C}(f_1) \arrow[r] \arrow[d, dotted] & A_1^\bullet[1] \arrow[d] \\ A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r] & \mathrm{C}(f_2) \arrow[r] & A_2^\bullet[1] \end{tikzcd}$$

can be completed with an arrow $\mathrm{C}(f_1) \to \mathrm{C}(f_2)$.

Proposition 1.36. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $\mathrm{C}(f)$ be its mapping cone with its natural arrows $\tau : B^\bullet \to \mathrm{C}(f)$ and $\pi : \mathrm{C}(f) \to A^\bullet[1]$. Then there exists a morphism of complexes*

$$g : A^\bullet[1] \to \mathrm{C}(\tau)$$

such that the following diagram commutes in $\mathrm{K}(\mathcal{A})$:

$$\begin{tikzcd} B^\bullet \arrow[r, "\tau"] \arrow[d, "="] & \mathrm{C}(f) \arrow[r, "\pi"] \arrow[d, "="] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dotted, "g"] & B^\bullet[1] \arrow[d, dotted, "="] \\ B^\bullet \arrow[r, "\tau"] & \mathrm{C}(f) \arrow[r, "\tau_\tau"] & \mathrm{C}(\tau) \arrow[r, "\pi_\tau"] & B^\bullet[1] \end{tikzcd}$$

Proof. We construct g on degree n as the arrow

$$A^\bullet[1]^n = A^{n+1} \longrightarrow \mathrm{C}(\tau)^n = B^{n+1} \oplus A^{n+1} \oplus B^n$$

defined by $(-f^{n+1}, \mathrm{Id}, 0)$. It clearly define a morphism of complexes. The inverse g^{-1} in $\mathrm{K}(\mathcal{A})$ can be given by the projection onto the middle factor (note that $g \circ g^{-1}$ is homotopic to identity, but not equal to identity in general). Now we let the reader check that the desired diagram is indeed commutative up to homotopy.

$\square$

Proposition 1.37. *Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism and $g : C^\bullet \to B^\bullet$ be an arbitrary morphism. Then there exists a commutative diagram in $\mathrm{K}(\mathcal{A})$:*

$$\begin{tikzcd} C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "qis"'] & B^\bullet. \end{tikzcd}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{C}(\tau \circ g)[-1] \arrow[r] \arrow[d, dotted] & C^\bullet \arrow[r] \arrow[d, "g"'] & \mathrm{C}(f) \arrow[r] \arrow[d, "="'] & \mathrm{C}(\tau \circ g) \arrow[d, dotted] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & \mathrm{C}(f) \arrow[r] & A^\bullet[1]. \end{tikzcd}$$

By the previous proposition, we know that $B^\bullet \xrightarrow{\tau} \mathrm{C}(f) \longrightarrow A^\bullet[1]$ is isomorphic (in $\mathrm{K}(\mathcal{A})$) to $B^\bullet \xrightarrow{\tau} \mathrm{C}(f) \longrightarrow \mathrm{C}(\tau)$, and thus it suffices to use the natural morphism $\mathrm{C}(\tau \circ g) \to \mathrm{C}(\tau)$ given by the identity on the second factor of $C^{n+1} \oplus \mathrm{C}(f)^n \to B^{n+1} \oplus \mathrm{C}(f)^n$.

Now define $C_0^\bullet := \mathrm{C}(\tau \circ g)[-1]$. Notice that $C_0^\bullet \to C^\bullet$ is a quasi-isomorphism. Indeed, since $A^\bullet \to B^\bullet$, we have that $H^n(\mathrm{C}(f)) = 0$ for all $n \in \mathbb{Z}$ (cf. Remark 1.35), and then applying the long exact sequence in cohomology to $\tau \circ g$ we get:

$$\cdots \longrightarrow H^n(C^\bullet) \longrightarrow H^n(\mathrm{C}(f)) \longrightarrow H^n(\mathrm{C}(\tau \circ g)) \longrightarrow H^{n+1}(C^\bullet) \longrightarrow \cdots.$$

But since $H^n(\mathrm{C}(f)) = 0$ we have that $H^n(\mathrm{C}(\tau \circ g)) \simeq H^{n+1}(C^\bullet)$ for all $n \in \mathbb{Z}$. $\square$

Remark 1.38.

- By construction, one can check that if g is also a quasi-isomorphism, so is $C_0^\bullet \to A^\bullet$.
- Acually, a dual statement holds: assume that we have a quasi-isomorphism $f : B^\bullet \to A^\bullet$ and any morphism $B^\bullet \to C^\bullet$. Then we can construc a commutative diagram:

$$\begin{tikzcd} B^\bullet \arrow[r, "f", "qis"'] \arrow[d] & A^\bullet \arrow[d] \\ C^\bullet \arrow[r, "qis"'] & C_0^\bullet. \end{tikzcd}$$

The proof is almost the same and is based on a dual statement of Proposition 1.36.

Corollary 1.39. *The composition of arrows in Definition 1.33 exists and is well-defined.*

Proof. It suffices to apply the previous proposition to the diagram :

$$\begin{tikzcd} C_1^\bullet \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \\ & B^\bullet & \end{tikzcd} .$$

The unicity (up to equivalence) is left to the reader. □

Remark 1.40. There is a natural functor $\mathcal{Q}_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ which is the identity on objects and which sends a (homotopy class of a) morphism $f : A^\bullet \to B^\bullet$ to the roof

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "f"] & \\ A & & B. \end{tikzcd}$$

Moreover, if f is a quasi-isomorphism, then $\mathcal{Q}_{\mathcal{A}}(f)$ is an isomorphism, which inverse is given by the roof

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "\mathrm{Id}"] & \\ B & & A. \end{tikzcd}$$

Remark 1.41. In $\mathrm{D}(\mathcal{A})$, an object $A^\bullet$ is isomorphic to 0 if and only if $H^n(A^\bullet) \simeq 0$ for all $n \in \mathbb{Z}$.

Definition 1.42. We say that a triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $\mathrm{K}(\mathcal{A})$ (resp. $\mathrm{D}(\mathcal{A})$) is distinguished if it is isomorphic in $\mathrm{K}(\mathcal{A})$ (resp. $\mathrm{D}(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} \mathrm{C}(f) \xrightarrow{\pi} A^\bullet[1],$$

where $f : A^\bullet \to B^\bullet$ is a morphism of complexes.

Proposition 1.43. *The natural shift functor $A^\bullet \to A^\bullet[1]$ and distinguished triangles given as in Definition 1.42 make the homotopy category of complexes $\mathrm{K}(\mathcal{A})$ and the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ into triangulated categories.*

Moreover, the natural functor $\mathcal{Q}_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is an exact functor of triangulated categories.

Proof. As the proof is long and technical, we refer to the literature ([7], I, 2 or [3], IV, 2). □

Remark 1.44. The idea behind the construction of the derived category is a procedure called *localization*: one constructs the localization of a category with respect to a localizing class of morphisms (which are quasi-isomorphisms in our case). The localized category can be made triangulated in a natural way if the localizing class of morphisms satisfies some conditions of *compatibility with triangulation*.

1.4 Properties of the derived category

The derived category $\mathrm{D}(\mathcal{A})$ is not abelian, but there still are some useful properties related to cohomology that we can prove.

Proposition 1.45. *Let $A, B, C \in \mathcal{A}$. We identify an object in $\mathcal{A}$ with its image under the full embedding $\mathcal{A} \to \mathrm{K}(\mathcal{A})$, i.e. with the associated complex concentrated in degree 0. If the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathcal{A}$ then the triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$$

is distinguished in $\mathrm{D}(\mathcal{A})$.

Proof. First, we need to define an arrow $C \to A[1]$. Notice that $\mathrm{C}(f)$ can be identified in this case with the complex $\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \longrightarrow \cdots$, with A in degree -1 and B in degree 0. Thus we can define the morphism of complexes $\mathrm{C}(f) \to C$ as :

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

In particular, this morphism is a quasi-isomorphism by exactness of the initial short exact sequence, and thus there is a inverse $C \to \mathrm{C}(f)$ in $\mathrm{D}(\mathcal{A})$. Thus one can define the arrow $\delta : C \to A[1]$ by composing the arrows $C \to \mathrm{C}(f)$ and the natural morphism $\mathrm{C}(f) \to A[1]$. We obtain the isomorphism of triangles (in $\mathrm{D}(\mathcal{A})$):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r, "g"] \arrow[d, "\mathrm{Id}"'] & C \arrow[r, "\delta"] \arrow[d] & A[1] \arrow[d, "\mathrm{Id}"'] \\
A \arrow[r, "f"] & B \arrow[r] & \mathrm{C}(f) \arrow[r] & A[1].
\end{tikzcd}$$

Note that these arrows are in $\mathrm{D}(\mathcal{A})$ so they should be thought of as roofs. ☐

Proposition 1.46. *Suppose $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}(\mathcal{A})$. Then there is a natural exact sequence*

$$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \cdots .$$

Proof. By definition of distinguished triangles, we have an isomorphism

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\simeq"] & B^\bullet \arrow[r] \arrow[d, "\simeq"] & C^\bullet \arrow[r] \arrow[d, "\simeq"] & A^\bullet[1] \arrow[d] \\
A_0^\bullet \arrow[r, "f"] & B_0^\bullet \arrow[r, "\tau"] & \mathrm{C}(f) \arrow[r, "\pi"] & A_0^\bullet[1],
\end{tikzcd}$$

where f is a morphism of complexes, and the vertical arrows are isomorphisms in $\mathrm{D}(\mathcal{A})$, *i.e.* quasi-isomorphisms. The sequence with the mapping cone induces a long exact sequence in cohomology (see Remark 1.35), and by the isomorphisms in cohomology we get:

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(A_0^\bullet) \arrow[r] \arrow[d, "\simeq"] & H^n(B_0^\bullet) \arrow[r] \arrow[d, "\simeq"] & H^n(\mathrm{C}(f)) \arrow[r] \arrow[d, "\simeq"] & H^n(A_0^\bullet[1]) \arrow[r] \arrow[d, "\simeq"] & \cdots \\
& H^n(A^\bullet) & H^n(B^\bullet) & H^n(C^\bullet) & H^{n+1}(A^\bullet) &
\end{tikzcd}$$

Since the first row is exact, up to composing with the isomorphisms, we get the desired natural exact sequence. ☐

In the following, we will consider the full subcategory $\mathrm{Kom}^*(\mathcal{A})$, with $* = +, -$ or b, consisting of complexes $A^\bullet$ with $A^n = 0$ for $n \ll 0, n \gg 0$ or $|n| \gg 0$ respectively. The same construction we performed before can be applied again to obtain the categories $\mathrm{K}^*(\mathcal{A})$ and $\mathrm{D}^*(\mathcal{A})$.

Proposition 1.47. *The natural functors $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$, where $* = +, -$ or b, define equivalences of $\mathrm{D}^*(\mathcal{A})$ with the full triangulated subcategories of all complexes $A^\bullet \in \mathrm{D}(\mathcal{A})$ with $H^n(A^\bullet) = 0$ for $n \ll 0, n \gg 0$ and $|n| \gg 0$ respectively.*

Proof. Suppose $H^n(A^\bullet) = 0$ for $n > n_0$. Then we have a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n_0-2} \arrow[r] \arrow[d, "="] & A^{n_0-1} \arrow[r] \arrow[d, "="] & \ker(d_1^{n_0}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A^{n_0-2} \arrow[r] & A^{n_0-1} \arrow[r] & A^{n_0} \arrow[r] & A^{n_0+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Thus $A^\bullet$ is isomorphic in $\mathrm{D}(\mathcal{A})$ to a complex bounded above, *i.e.* a complex in $\mathrm{D}^-(\mathcal{A})$. Similarly, if $H^n(A^\bullet) = 0$ for $n < n_0$ one considers :

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n_0-1} \arrow[r] \arrow[d] & A^{n_0} \arrow[r] \arrow[d] & A^{n_0+1} \arrow[r] \arrow[d, "="] & A^{n_0+2} \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{coker}(d^{n_0-1}) \arrow[r] & A^{n_0+1} \arrow[r] & A^{n_0+2} \arrow[r] & \cdots .
\end{tikzcd}$$

In the case $* =$ b, one can use both sequences combined. These prove that the functors are essentially surjective, and they are clearly fully faithful, so the proof is finished. ☐

Before going to the next section, we need to define the notion of *injective* and *projective* resolutions, which will be useful when we will need to extend functors $F : \mathcal{A} \to \mathcal{B}$ into functors $RF : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$.

Definition 1.48. Let $\mathcal{A}$ be an abelian category.

- An object $I \in \mathcal{A}$ (resp. $P \in \mathcal{A}$) is said *injective* (resp. *projective*) if the functor $Hom(\ \ , I)$ is exact (resp. $Hom(P, \ \ )$ is exact).
- We say that the category $\mathcal{A}$ contains *enough injective* (resp. *enough projectives*) objects if for any object $A \in \mathcal{A}$ there exists an injective morphism $A \to I$ with I injective (resp. a surjective morphism $P \to A$ with P projective).
- An *injective resolution* of an object $A \in \mathcal{A}$ is an exact sequence

  $$0 \to A \to I^0 \to I^1 \to \cdots$$

  with all I^n injective. Similarly, a *projective* resolution of A consists in an exact sequence

  $$\cdots \to P^{-1} \to P^0 \to A \to 0$$

  with all P^n projective.

Remark 1.49. • Since the functors $Hom(\ \ , I)$ and $Hom(P, \ \ )$ are left-exact for any objects I and P, injectivity or projectivity can easily be described as follows.

An object $I \in \mathcal{A}$ is injective if for any injective arrow $A \hookrightarrow B$ and any arrow $A \to I$, there exists an arrow $B \to I$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr] & & B \arrow[dl, dotted] \\
& I &
\end{tikzcd}$$

Similarly, an object $P \in \mathcal{A}$ is projective if for any surjective arrow $B \twoheadrightarrow C$ and any arrow $P \to C$, there exist an arrow $P \to B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[rr, two heads] & & C \\ & P \arrow[ul, dotted] \arrow[ur] & \end{tikzcd}$$

- One notices that the datum of an injective resolution $I^\bullet$ of A is equivalent to the datum of a quasi-isomorphism $A \to I^\bullet$ with $I^n = 0$ for $n < 0$ and all I^n injective. Similarly, a projective resolution $P^\bullet$ of A is the datum of a quasi-isomorphism $P^\bullet \to A$ with $P^n = 0$ for all $n > 0$ and all P^n projective.

Proposition 1.50. *Suppose that $\mathcal{A}$ is a category with enough injectives. For any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$, there exist a complex $I^\bullet \in \mathrm{K}^+(\mathcal{A})$ with $I^n \in \mathcal{A}$ injective $\forall n \in \mathbb{Z}$ and a quasi-isomorphism $A^\bullet \to I^\bullet$.*

Proof. We prove it by induction. Since $A^\bullet$ is bounded below, we may assume that it is of the shape

$$0 \to A^0 \to A^1 \to \cdots .$$

By assumption, there exists an injective object I^0 and an injective arrow $A^0 \to I^0$. The induced arrow $A^\bullet \to (I^0 \to 0 \to \cdots)$ has the property that $H^n(f_0)$ is an isomorphism for $n < 0$ and is injective for $n = 0$.

Suppose we have constructed a morphism

$$f_i : A^\bullet \longrightarrow (\cdots \to I^{i-1} \to I^i \to 0 \to \cdots)$$

such that $H^n(f_i)$ is an isomorphism for all $n < i$ and injective for $i = n$, and such that all I^n are injective. Then choose an injective object I^{i+1} containing $B^{i+1} := ((I^i/I^{i-1}) \oplus A^{i+1})/A^i$,. We have the arrows $I^i \to I^{i+1}$ given by composing the arrows $I^i \to I^i/I^{i-1}$, $I^i/I^{i-1} \to B^{i+1}$ and $B^{i+1} \hookrightarrow I^{i+2}$; and $A^i \to I^{i+1}$ in a similar way. We obtain the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "f_{i+1}^{i-1}"] & A^i \arrow[r] \arrow[d, "f_{i+1}^{i}"] & A^{i+1} \arrow[r] \arrow[d, "f_{i+1}^{i+1}"] & A^{i+2} \arrow[r] \arrow[d, "f_{i+1}^{i+2}"] & \cdots \\ \cdots \arrow[r] & I^{i-1} \arrow[r] & I^i \arrow[r] & I^{i+1} \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

Now we just have to check that all induction properties are satisfied:

- First, the second row is still a complex by construction of I^{i+1}.
- The middle square is commutative. Indeed, both composition $A^i \to A^{i+1} \to I^{i+1}$ and $A^i \to I^i \to I^{i+1}$ are trivial by construction of B^{i+1}.
- The new map $H^i(f_{i+1})$ is now bijective: it was injective when we took $I^i/d(I^{i-1})$ as target space, but now we restrict the latter to $\ker(I^i \to I^{i+1})$. But this kernel is composed by the image of I^{i+1} (which vanishes in cohomology) and the image of A^i, thus it implies surjectivity of $H^i(f_{i+1})$.
- The last thing to check is that $H^{i+1}(f_{i+1})$ is injective. But it is quite clear since up to take the quotient with respect to A^i, the map $A^{i+1} \to I^{i+1}/I^i$ is injective.

□

Corollary 1.51. *Let $\mathcal{A}$ be an abelian category with enough injectives. Any object $A^\bullet \in \mathrm{D}(\mathcal{A})$ with $H^n(A^\bullet) = 0$ for $n \ll 0$ is isomorphic in $\mathrm{D}(\mathcal{A})$ to a complex $I^\bullet$ of injective objects with $I^n = 0$ for $n \ll 0$.*

A dual statement of Proposition 1.50 is true in a category with enough projectives, considering $\mathrm{K}^-(\mathcal{A})$ instead of $\mathrm{K}^+(\mathcal{A})$: for any $A^\bullet \in \mathrm{K}^-(\mathcal{A})$ there exists a complex $P^\bullet \in \mathrm{K}^-(\mathcal{A})$ with $P^n \in \mathcal{A}$ projective objects and a quasi-isomorphism $P^\bullet \to A^\bullet$.

A proof of the next two technical lemmas can be found in ([9], 2.38 and 2.39).

Lemma 1.52. *Suppose $A^\bullet \to B^\bullet$ is a quasi-isomorphism between two complexes $A^\bullet, B^\bullet \in \mathrm{K}^+(\mathcal{A})$. Then for any complex $I^\bullet$ of injectives objects with $I^n = 0$ for $n \ll 0$ the induced map*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(B^\bullet, I^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet)$$

is bijective.

Lemma 1.53. *Let $A^\bullet, I^\bullet \in \mathrm{K}^+(\mathcal{A})$ such that all I^n are injective. Then*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, I^\bullet).$$

For the next proposition, consider the full additive subcategory $\mathcal{I} \subset \mathcal{A}$ of all injectives of an abelian category $\mathcal{A}$: we can construct as before the homotopy category $\mathrm{K}^*(\mathcal{I})$ and the functor $\mathcal{Q}_\mathcal{A}$ induces a natural exact functor $\iota : \mathrm{K}^*(\mathcal{I}) \to \mathrm{D}^*(\mathcal{A})$.

Proposition 1.54. *Suppose that $\mathcal{A}$ contains enough injectives. Then the natural functor*

$$\iota : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$$

is an equivalence.

Proof. The functor is fully faithful. Indeed, let $I^\bullet, J^\bullet$ be two complexes in $\mathrm{K}^+(\mathcal{I})$. Since $\mathcal{I}$ is a full subcategory and by the previous lemma, we have

$$\mathrm{Hom}_{\mathrm{K}^+(\mathcal{I})}(I^\bullet, J^\bullet) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(I^\bullet, J^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}^+(\mathcal{A})}(I^\bullet, J^\bullet).$$

To see that the functor is also essentially surjective, one applies Proposition 1.50. □

2 Derived functors

2.1 Derived functors

In this section, the main goal will be to lift functors between abelian categories (or homotopy categories) to functors between the associated derived categories.

Lemma 2.1. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $F : \mathrm{K}^*(\mathcal{A}) \to \mathrm{K}^*(\mathcal{B})$ be an exact functor of triangulated categories. Then F naturally induces a commutative diagram :*

$$\begin{tikzcd}
\mathrm{K}^*(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{K}^*(\mathcal{B}) \arrow[d] \\
\mathrm{D}^*(\mathcal{A}) \arrow[r] & \mathrm{D}^*(\mathcal{B})
\end{tikzcd}$$

if one of the following equivalent conditions holds true:

1. *A quasi-isomorphism is mapped by F to a quasi-isomorphism.*
2. *The image of an acyclic complex is acyclic.*

Proof. First let's show that these two conditions are equivalent. The step $1 \Rightarrow 2$ is obvious. To see $2 \Rightarrow 1$, consider a morphism of complexes $f : A^\bullet \to B^\bullet$, then the triangle

$$A^\bullet \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1]$$

is distinguished, and $\mathrm{C}(f)$ is acyclic if and only if f is a quasi-isomorphism (cf. Remark 1.35). But since F is exact and additive, $F(f)$ is a quasi-isomorphism if and only if $\mathrm{C}(F(f)) = F(\mathrm{C}(f))$ is acyclic.

Assume that 1 is satisfied. The functor F can easily be lift up to the derived categories: an object $A^\bullet$ is mapped to $F(A^\bullet)$, viewed as objects in the derived categories, and a roof

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

is mapped to the roof

$$\begin{tikzcd} & F(C^\bullet) \arrow[dl, "qis"'] \arrow[dr] & \\ F(A^\bullet) & & F(B^\bullet) \end{tikzcd}$$

□

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and assume that $\mathcal{A}$ contains enough injectives. The functor F induces a functor $\mathrm{K}(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ sending a complex $(A^n)_{n\in\mathbb{Z}}$ to $(F(A^n))_{n\in\mathbb{Z}}$, and a morphism of complexes $(f^n)_{n\in\mathbb{Z}}$ to $(F(f^n))_{n\in\mathbb{Z}}$. The latter makes sense in the homotopy categories: if h is a homotopy between to morphisms of complexes f and g, then $F(h)$ is a homotopy between $F(f)$ and $F(g)$ since F is additive.

We have the equivalence $\iota : \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$, so we can consider a quasi-inverse ι^{-1} of ι by choosing a complex of injective objects quasi-isomorphic to any given complex that is bounded below. We obtain the diagram:

$$\begin{tikzcd} \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] \arrow[dr, "\iota"] & \mathrm{K}^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "\mathcal{Q}_\mathcal{A}"] & \mathrm{K}^+(\mathcal{B}) \arrow[d, "\mathcal{Q}_\mathcal{B}"] \\ & \mathrm{D}^+(\mathcal{A}) \arrow[ul, bend left, "\iota^{-1}"] & \mathrm{D}^+(\mathcal{B}). \end{tikzcd}$$

Definition 2.2. The *right derived functor* of F is the functor:

$$RF := \mathcal{Q}_\mathcal{B} \circ \mathrm{K}(F) \circ \iota^{-1} : \mathrm{D}^+(\mathcal{A}) \longrightarrow \mathrm{D}^+(\mathcal{B}).$$

In other words, the right derived functor consists in replacing a complex by a complex of injectives, applying $K(F)$ and embedding it into the target derived category.

Proposition 2.3. *1. There exists a natural morphism of functors*

$$\mathcal{Q}_\mathcal{B} \circ \mathrm{K}(F) \longrightarrow RF \circ \mathcal{Q}_\mathcal{A}.$$

2. The right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to D^+(\mathcal{B})$ is an exact functor of triangulated categories.

Proof. 1. Let $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ and $I^\bullet := \iota^{-1}(A^\bullet)$. The natural transformation $\mathrm{Id} \to \iota \circ \iota^{-1}$ yields a functorial morphism $A^\bullet \to I^\bullet$ in $D^+(\mathcal{A})$. This morphism is given by a roof $A^\bullet \leftarrow C^\bullet \to I^\bullet$, but since $I^\bullet$ is injective it yields to a unique morphism $A^\bullet \to I^\bullet$ in $\mathrm{K}(\mathcal{A})$

by Lemma 1.52. Notice that this morphism is independent on the choice of $C^\bullet$: assume we have two equivalent roofs

$$\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet \arrow[rr, dotted, "t_C"'] & & I^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet \arrow[rr, dotted] & & I^\bullet, \end{tikzcd}$$

then it means that we have the commutative diagram

$$\begin{tikzcd} & E^\bullet \arrow[dl] \arrow[dr] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ C^\bullet \arrow[d] \arrow[dr] & & D^\bullet \arrow[d] \arrow[dl] \\ A^\bullet \arrow[r, shift left] \arrow[r, shift right, "t_C,\ t_D"'] & & I^\bullet. \end{tikzcd}$$

We obtain the equalities $g = t_C \circ f$ and $g = t_D \circ f$. But f is a quasi-isomorphism, so by Lemma 1.52 there is a unique map $j : A^\bullet \to I^\bullet$ such that $g = j \circ f$. Thus we get $t_C = t_D$. Finally, we obtain a functorial morphism

$$\mathrm{K}(F(A^\bullet)) \to \mathrm{K}(F(I^\bullet)) = RF(A^\bullet).$$

2. The category $K^+(\mathcal{I}_\mathcal{A})$ is triangulated: if $f : I^\bullet \to J^\bullet$ is a morphism of complexes between complexes of injective objects, then $\mathrm{C}(f)$ is also a complex of injective objects. The functor $\iota : \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$ is clearly an exact functor (between triangulated categories), and thus ι^{-1} is also exact (cf. Proposition 1.16). Moreover, $\mathrm{K}(F)$ is exact: F is additive, so F preserves mapping cones. Finally, since $\mathcal{Q}_\mathcal{B}$ is exact, we obtain that RF is the composition of three exact functors and, therefore, is itself exact.

□

Definition 2.4. Let $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ be the right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ we define:

$$R^iF(A^\bullet) := H^i(RF(A^\bullet)) \in \mathcal{B}.$$

Remark 2.5. • If A is a complex concentrated in degree 0, then we can give a more precise description of $R^iF(A)$. Indeed, consider an injective resolution $I^\bullet$ of A, *i.e.* an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots .$$

We obtain that $R^iF(A) = H^i(F(I^\bullet))$, and in particular we have $R^0F(A) = F(A)$.

• Any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ gives rise to a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \cdots \to R^nF(B) \to R^nF(C) \to R^{n+1}F(A) \to \cdots .$$

Indeed, the exact sequence in $\mathcal{A}$ gives rise to a distinguished triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ by Proposition 1.45, then it suffices to apply Proposition 1.46 to conclude.

• All the constructions we made could have been performed in the dual way: if you consider a functor F which is right exact, $\mathrm{K}(F) : \mathrm{K}^-(\mathcal{A}) \to \mathrm{K}^-(\mathcal{B})$ and then define the *left derived functor* LF by applying $\mathrm{K}(F)$ to a complex $P^\bullet$ of *projective* objects quasi-isomorphic to $A^\bullet$.

Now we will give a generalization of the construction of the right derived functor.

Proposition 2.6. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $F : \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ an exact functor. Suppose there exists a triangulated subcategory $\mathcal{K}_F \subset \mathrm{K}^+(\mathcal{A})$ which is adapted to F, i.e. which satisfies the following two conditions:*

1. *If $A^\bullet \in \mathcal{K}_F$ is acyclic, then $F(A^\bullet)$ is acyclic.*
2. *Given any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ there is an object $T_{A^\bullet} \in \mathcal{K}_F$ and a quasi-isomorphism $A^\bullet \to T_{A^\bullet}$.*

Then there exists a right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ satisfying the properties of Proposition 2.3.

Proof. The functor RF is defined as follows:

- Let $A^\bullet \in \mathrm{D}^+(\mathcal{A})$. There is a quasi-isomorphism $A^\bullet \to T_{A^\bullet}$ for some $T_{A^\bullet} \in \mathcal{K}_F$. Then define $RF(A^\bullet) := F(T_{A^\bullet})$.
- Let $A^\bullet, B^\bullet \in \mathrm{D}^+(\mathcal{A})$. Consider an arrow $A^\bullet \to B^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ given by a roof $A^\bullet \leftarrow C^\bullet \to B^\bullet$. In $\mathrm{K}^+(\mathcal{A})$ we have the diagram

$$\begin{tikzcd} C^\bullet \arrow[r, "qis"] \arrow[d, "qis"'] & T_{C^\bullet}, \\ T_{A^\bullet} & \end{tikzcd}$$

and it can be completed in $\mathrm{K}^+(\mathcal{A})$ into the diagram

$$\begin{tikzcd} C^\bullet \arrow[r, "qis"] \arrow[d, "qis"'] & T_{C^\bullet} \arrow[d, "qis"] \\ T_{A^\bullet} \arrow[r, "qis"'] & D^\bullet_A. \end{tikzcd}$$

Composing with the quasi-isomorphism $D^\bullet_A \to T_{D^\bullet_A}$, and doing the same with $B^\bullet$, we obtain a roof

$$\begin{tikzcd} & T_{C^\bullet} \arrow[dl, "qis"'] \arrow[dr] & \\ T_{D^\bullet_A} & & T_{D^\bullet_B}. \end{tikzcd}$$

Since $T_{D^\bullet_A}$ and $T_{A^\bullet}$ are quasi-isomorphic within $\mathcal{K}_F$, so are their images by the functor F (cf. proof of Lemma 2.1), and the same holds with $B^\bullet$. Thus they define isomorphic objects in $\mathrm{D}(\mathcal{B})$. We defined the image of our initial arrow $A^\bullet \to B^\bullet$ in $D^+(\mathcal{A})$ by RF as the arrow given by the roof

$$\begin{tikzcd} & F(T_{C^\bullet}) \arrow[dl, "qis"'] \arrow[dr] & \\ F(T_{A^\bullet}) & & F(T_{B^\bullet}). \end{tikzcd}$$

$\square$

Corollary 2.7. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor (here $\mathcal{A}$ might not contain enough injectives), and assume that there exists a subclass of objects $\mathcal{I}_F \subset \mathcal{A}$ which are F-adapted, i.e. which is stable by finite sum and such that:*

1. *If $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ is acyclic with all $A^\bullet \in \mathcal{I}_F$, then $F(A^\bullet)$ is acyclic.*

2. Any object in $\mathcal{A}$ can be embedded into an object of $\mathcal{I}_F$.

Then there exists a right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to D(\mathcal{B})$ satisfying the properties of Proposition 2.3.

Proof. It suffices to check that the subcategory $\mathcal{K}_F \subset \mathrm{K}^+(\mathcal{A})$ defined as the full subcategory of complexes of objects in $\mathcal{I}_F$ satisfies the hypothesis of Proposition 2.6.

First, since $\mathcal{I}_F$ is stable by finite sum, $\mathcal{K}_F$ contains all mapping cones of morphism between any complexes in it. Then, by the Proposition 1.18, $\mathcal{K}_F$ is indeed a triangulated subcategory of $K^+(\mathcal{A})$. Now, we just need to check the two conditions of the theorem. The first condition is obvious, and the second conditions can be proved by the same proof given for Proposition 1.50 (the latter does not use injectivity of objects involved!). $\square$

A similar construction could have been done, once again, with right exact functors. As before, we would have asked any object A in $\mathcal{A}$ to fit in an exact sequence $P \to A \to 0$ for some $P \in \mathcal{I}_F$.

Definition 2.8. Let $A \in \mathcal{A}$ be an object in an abelian category containing enough injectives. Then we defined

$$\mathrm{Ext}^n(A, \ \ ) := H^n \circ R\,\mathrm{Hom}(A, \ \ ).$$

Proposition 2.9. *Suppose $A, B \in \mathcal{A}$ are objects of an abelian category containing enough injectives. Then for all $n \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathrm{Ext}^n(A, B) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n]).$$

Proof. Notice that here we identify once again objects in $\mathcal{A}$ with complexes concentrated in degree 0. Consider an injective resolution $B \to I^0 \to I^1 \to \cdots$, then $R\,\mathrm{Hom}(A, B) \simeq (\mathrm{Hom}(A, I^n))_{n\in\mathbb{N}}$. Now $f \in \mathrm{Hom}(A, I^n)$ is the kernel of $\mathrm{Hom}(A, I^n) \to \mathrm{Hom}(A, I^{n+1})$ if and only if it defines a morphism of complexes $f : A \to I^\bullet[n]$. Such a morphism is (homotopically) trivial if and only if f is in the image of $\mathrm{Hom}(A, I^{n-1}) \to \mathrm{Hom}(A, I^n)$. These last claims reads on the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] \arrow[dl, dotted] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Then $\mathrm{Ext}^n(A, B) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, I^\bullet[n]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, I^\bullet[n])$ since $I^\bullet$ is a complex of injectives. $\square$

Remark 2.10. The name "Ext" comes from "Extensions" because $\mathrm{Ext}^1(A, B)$ is in bijection with the set of extension $0 \to B \to L \to A \to 0$ in $\mathcal{A}$. Indeed, consider an element in $\mathrm{Ext}^1(A, B) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[1])$ given by a roof $A \leftarrow L^\bullet \to B[1]$. Assume that $L^i = 0\ \forall i > 0$ by replacing L^0 by $\ker(d_L^0)$ if necessary. Then we have the extension

$$0 \to B \to \frac{L^0 \oplus B}{L^{-1}} \to A \to 0.$$

Conversely, if $0 \to B \to L \to A \to 0$ is an extension in $\mathcal{A}$, then define $L^\bullet$ by $L^{-1} = B$, $L^0 = L$ and all other L^i's trivial. Then we get a roof $A \leftarrow L^\bullet \to B[1]$.

Definition 2.11. Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ and $B^\bullet \in \mathrm{K}^+(\mathcal{A})$. We defined the *inner hom* $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ as the complex

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(A^k, B^{k+n})$$

with differentials $d((f_k)_{k\in\mathbb{Z}}) := d_B \circ f_k - (-1)^n f_{k+1} \circ d_A$.

Proposition 2.12. *Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ be a complex of objects in a abelian category containing enough injectives. The the right derived functor*

$$R\,\mathrm{Hom}^\bullet(A^\bullet, \ \ ) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathbf{Ab})$$

exists, and if we set $\mathrm{Ext}^n(A^\bullet, B^\bullet) := H^n(R\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$ *we have*

$$\mathrm{Ext}^n(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[n]).$$

Proof. To prove the existence of $R\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$, one checks that the full triangulated subcategory of $\mathrm{K}^+(\mathcal{A})$ of complexes of injectives objects is adapted to this functor. The second statement follows from arguments of the proof of Proposition 2.9 adapted to this more general situation. □

Remark 2.13. If the abelian category $\mathcal{A}$ has also enough projectives, then we obtain a bifunctor

$$R\,\mathrm{Hom}(\ \ ,\ \ ) : \mathrm{D}^-(\mathcal{A})^{op} \times \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathbf{Ab}).$$

If $\mathcal{A}$ has only enough injectives, we still can defined a derived functor $R\,\mathrm{Hom}(\ \ , B^\bullet) : \mathrm{D}^-(\mathcal{A})^{op} \to \mathrm{D}(\mathbf{Ab})$ if $B^\bullet$ is bounded below.

Before going on a next section, we give a last result on derived functor which will be really useful when considering composition of functors.

Proposition 2.14. *Let $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ be two left exact functors between abelian categories. Assume that there exist adapted classes $\mathcal{I}_{F_1} \subset \mathcal{A}$ and $\mathcal{I}_{F_2} \subset \mathcal{B}$ for F_1 and F_2 respectively such that $F(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$.*

Then the derived functor $R(F_2 \circ F_2) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{C})$ exists and there is a natural isomorphism

$$R(F_2 \circ F_1) \simeq RF_2 \circ RF_1.$$

Proof. The existence of RF_1 and RF_2 are provided by the assumptions, and since $\mathcal{I}_{F_1}$ is adapted to $F_2 \circ F_1$, $R(F_2 \circ F_1)$ exists aswell. The natural isomorphism is given by the following remark. Let $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ be isomorphic to $I^\bullet \in \mathrm{K}^+(\mathcal{I}_{F_1})$, then

$$\begin{aligned} R(F_2 \circ F_1)(A^\bullet) &\simeq \mathrm{K}(F_2 \circ F_1)(I^\bullet), \\ &\simeq (\mathrm{K}(F_2) \circ \mathrm{K}(F_1))(I^\bullet), \\ &\simeq \mathrm{K}(F_2)(\mathrm{K}(F_1)(I^\bullet)), \\ &\simeq RF_2(\mathrm{K}(F_1)(I^\bullet)), \\ &\simeq RF_2(RF_1(A^\bullet)). \end{aligned}$$

□

2.2 Spectral sequences

We consider an abelian category $\mathcal{A}$.

Definition 2.15. A *spectral sequence* is the data of a collection of objects

$$(E_r^{p,q}, E^n),\ n, p, q, r \in \mathbb{Z},\ r \geq 1,$$

and arrow

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

satisfying the next four conditions:

1. $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p, q, r.

2. There are isomorphisms
$$E_{r+1}^{p,q} \simeq H^0(E_r^{p+\bullet r,q-\bullet r+\bullet}).$$

3. For any (p,q) there exists an $r_0(p,q)$ such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for al $r \geq r_0$.

4. There is a decreasing filtration
$$\cdots \subset F^{p+1}E^n \subset F^pE^n \subset \cdots \subset E^n$$
such that
$$\bigcap F^pE^n = 0 \text{ and } \bigcup F^pE^n = E^n,$$
and isomorphisms
$$E_{r_0(p,q)}^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}.$$

Remark 2.16.
- For all $r \geq r_0(p,q)$, we have $E_r^{p,q} \simeq E_{r_0(p,q)}^{p,q}$. Usually we will denoted $E_\infty^{p,q} := E_{r_0(p,q)}^{p,q}$.

- When the objects of a spectral sequence is given on a layer (*i.e.* for a given $r \geq 1$), then the next ones can be deduced (up to isomorphism) from the property 2. Thus we will often introduce a spectral sequence writing
$$E_r^{p,q} \Rightarrow E^{p+q}$$
for a given r. In most cases, it will be given for $r = 2$.

- Let's give an example of spectral sequence. For instance, assume that all objects considered are (finite dimensional) vector spaces, and that all differentials on layer 2 vanish for some reason. Then, for all p,q, $E_2^{p,q} = E_\infty^{p,q}$. Then $E_2^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}$ yields
$$F^pE^n = E_2^{p,n-p} \oplus F^{p+1}E^n = E_2^{p,n-p} \oplus E^{p+1,n-p-1} \oplus F^{p+2}E^n = \cdots .$$
Thus $F^pE^n = \bigoplus_{k\geq 0} E^{p+k,n-p-k}$, and we obtain
$$E^n = \bigcup F^pE^n = \bigoplus_{k\in\mathbb{Z}} E_2^{k,n-k}.$$

- If we just know that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for some fixed p and q, *i.e.* $E^{p,q} = E_\infty^{p,q}$, then
$$E_r^{p,q} \neq 0 \Longrightarrow E^{p+q} \neq 0$$
since $0 \neq E_r^{p,q} = F^pE^{p+q}/F^{p+1}E^{p+q}$.

- If $E_r^{p,q} = 0$ for all p,q for a given r, then $0 = F^pE^{p+q}/F^{p+1}E^{p+q}$ thus $F^pE^{p+q} = F^{p+1}E^{p+q}$ for all p,q. But then
$$E^{p+q} = \bigcup F^pE^{p+q} = F^0E^q,$$
$$0 = \bigcap F^pE^{p+q} = F^0E^q,$$
and thus we can conclude that $E^n = 0$ for all $n \in \mathbb{Z}$.

Definition 2.17. A *double complex* $K^{\bullet,\bullet}$ consists of objects $K^{i,j}$ for $i,j \in \mathbb{Z}$ and morphisms
$$d_I^{i,j} : K^{i,j} \to K^{i+1,j} \text{ and } d_{II}^{i,j} : K^{i,j} \to K^{i,j+1}$$
satisfying
$$d_I^2 = d_{II}^2 = d_Id_{II} + d_{II}d_I = 0.$$
The *total complex* $K^\bullet := \text{tot}(K^{\bullet,\bullet})$ of the double complex is the complex $K^n = \bigoplus_{i+j=n} K^{i,j}$ with differentials $d = d_I + d_{II}$.

The complex $K^\bullet = \mathrm{tot}(K^{\bullet,\bullet})$ is naturally endowed with a decreasing filtration

$$F^l K^n := \bigoplus_{j\geq l} K^{n-j,j},$$

which satisfies $d_I(F^l K^n) \subset F^l(K^{n+1})$.

We will write $H_I^n(K^{\bullet,\bullet})$ for the complex given by $(H^n(K^{\bullet,q}))_{q\in\mathbb{Z}}$, and similarly $H_{II}^n(K^{\bullet,\bullet}) := (H_{II}^n(K^{q,\bullet}))_{q\in\mathbb{Z}}$.

Proposition 2.18. *Suppose $K^{\bullet,\bullet}$ is a double complex such that for any n one has $K^{n-l,l} = 0$ for $|l| \gg 0$. Then there is a spectral sequence:*

$$E_2^{p,q} = H_{II}^p H_I^q(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(K^\bullet).$$

The proof is based on the structure of *filtred complex* of $\mathrm{tot}(K^{\bullet,\bullet})$: a complex $A^\bullet$ is a *filtred complex* if it admits a decreasing filtration

$$\cdots F^l A^n \subset F^{l-1}A^n \subset \cdots \subset A^n$$

for every object A^n such that $d(F^l A^n) \subset F^l A^{n+1}$.

Proof. See ([3], III, §7, 5). ☐

Definition 2.19. Let $A^\bullet$ be a complex in $\mathrm{K}^+(\mathcal{A})$. A *Cartan-Eilenberg resolution* of $A^\bullet$ is a double complex $C^{\bullet,\bullet}$ together with a morphism of complexes $A^\bullet \to C^{\bullet,0}$ such that :

1. $C^{i,j} = 0$ for $j < 0$.
2. The sequences
$$A^n \to C^{n,0} \to C^{n,1} \to \cdots$$
are injective resolutions of A^n, and the induced sequences
$$\ker(d_A^n) \to \ker(d_I^{n,0}) \to \ker(d_I^{n,1}) \to \cdots$$
$$\mathrm{Im}(d_A^n) \to \mathrm{Im}(d_I^{n,0}) \to \mathrm{Im}(d_I^{n,1}) \to \cdots$$
$$H^n(A^\bullet) \to H_I^n(C^{\bullet,0}) \to H_I^n(C^{\bullet,1}) \to \cdots$$
are injective resolutions of $\ker(d_A^n)$, $\mathrm{Im}(d_A^n)$ and $H^n(A^\bullet)$ respectively.
3. All the short exact sequences
$$0 \to \ker(d_I^{i,j}) \to C^{i,j} \to \mathrm{Im}(d_I^{i,j}) \to 0$$
split.

Lemma 2.20. *If $\mathcal{A}$ has enough injectives, then any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

Proof. See ([3], §7, 11). ☐

Proposition 2.21. *Let $F_1 : \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$ and $F_2 : \mathrm{K}^+(\mathcal{B}) \to \mathrm{K}(\mathcal{C})$ be two exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ contain enough injectives and that the image under F_1 of a complex $I^\bullet \in \mathrm{K}^+(\mathcal{A})$ of injective objects is contained in a F_2-adapted triangulated subcategory $\mathcal{K}_{F_2}$ of $\mathrm{K}^+(\mathcal{B})$.*

Then for any complex $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ there exists a spectral sequence

$$E_2^{p,q} = R^p F_2(R^q F_1(A^\bullet)) \Rightarrow E^n = R^n(F_2 \circ F_1)(A^\bullet).$$

Proof. First assume that the proposition is true for $F_1 = \mathrm{Id}$, *i.e.* we have the spectral sequence

$$E_2^{p,q} = R^p F_2(H^q(A^\bullet)) \Rightarrow E^n = R^n(F_2)(A^\bullet).$$

By construction of derived functors we have

$$R^p F_2(R^q F_1(A^\bullet)) = R^p(H^q(F_1(I^\bullet)))$$

for some complex of injective objects $I^\bullet$ quasi-isomorphic to $A^\bullet$, and since

$$R^n(F_2 \circ F_1)(A^\bullet) = H^n(F_2 \circ F_1(I^\bullet)) = R^n F_2(F_1(I^\bullet)),$$

the general case is also true.

Thus it suffices to show the proposition with $F_1 = \mathrm{Id}$. We will write $F := F_2$. Consider a Cartan-Eilenberg resolution $C^{\bullet,\bullet}$ of $A^\bullet$ and set $K^{\bullet,\bullet} := F(C^{\bullet,\bullet})$. Since F is additive, it preserves direct sums, and since $C^{i,j} \simeq \ker d_I^{i,j} \oplus \mathrm{Im}\, d_I^{i,j}$ we have $H_I^q(K^{\bullet,p}) = F H_I^q(C^{\bullet,p})$. But fixing q and running p, $H_I^q(C^{\bullet,p})$ defines an injective resolution of $H^q(A^\bullet)$ and thus we obtain

$$H_{II}^p H_I^q(K^{\bullet,\bullet}) = R^p F(H^q(A^\bullet)).$$

Applying the spectral sequence defined in Proposition 2.18, and using the fact that $A^\bullet \to \mathrm{tot}(C^{\bullet,\bullet})$ is a quasi-isomorphism, we find the limit

$$\begin{aligned} H^{p+q}(\mathrm{tot}(K^{\bullet,\bullet})) &= H^{p+q}(F(\mathrm{tot}(C^{\bullet,\bullet}))), \\ &= H^{p+q}(RF(A^\bullet)), \\ &= R^{p+q} F(A^\bullet). \end{aligned}$$

$\square$

Part II
Algebraic geometry

In this part, we introduce basic notions in algebraic geometry. We use the language of schemes but we try to skip scheme-theoretic properties as we focus on the notion that will be used in the last part. More content can be found in [8] and [4].

3 Varieties and schemes

3.1 Varieties and morphisms

Let k be a field, A be the ring $k[x_1, \dots, x_n]$. We denote by $\mathbb{A}^n_k$ (or $\mathbb{A}^n$ when the context is clear) the *affine n-space* k^n endowed with the *Zariski topology* in which closed subsets are *algebraic subsets*, *i.e.* the ones of the form

$$Z(T) := \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in T\}$$

for some subset $T \subseteq A$.

Definition 3.1. Irreducible closed subsets of $\mathbb{A}^n$ (with the induced topology) are called *affine varieties*. Open subsets of an affine variety, endowed with the induced topology, are called *quasi-affine varieties*.

Given an affine variety $V \subseteq \mathbb{A}^n$, the set $I(V) \subseteq A$ is the ideal of A given by all polynomials vanishing on V. Then we define the affine coordinate ring of V as the quotient ring $A/I(V)$, and the *dimension* of V as the (Krull) dimension of $A/I(V)$.

In this text, we will be more interested in the notion of projective variety. Let $\mathbb{P}^n$ be the usual projective n-space. Let $S = k[x_0, \dots, x_n]$ be the usual polynomial ring considered as a graded ring (with the natural graduation $\deg(x_i) = 1$). For a subset T of homogeneous elements in S, set

$$Z(T) := \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in T\},$$

(the condition $f = 0$ is well defined in $\mathbb{P}^n$ for f homogeneous). A subset $Y \subseteq \mathbb{P}^n$ is called *algebraic* if there exists a set T of homogeneous elements in S such that $Y = Z(T)$. We defined the *Zariski topology* on $\mathbb{P}^n$ by defining algebraic sets as closed subsets.

Definition 3.2. A *projective variety* is an irreducible closed subset of $\mathbb{P}^n$ (with the induced topology). An open subset of a projective variety, endowed with the induced topology, is a *quasi-projective* variety.

The dimension of a (quasi-)projective variety is defined as its dimension as a topological space, *i.e.* the supremum of all integers n such that there exist a chain $Z_0 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets.

Proposition 3.3. *A projective (resp. quasi-projective) variety admits a finite cover by open subsets homeomorphic to affine (resp. quasi affine) varieties.*

Proof. It is enough to show that $\mathbb{P}^n$ admits a cover by opens U_i's homeomorphic to $\mathbb{A}^n$, then any projective variety $V \subseteq \mathbb{P}^n$ admits a cover $V \cap U_i$).

Consider the subset $H_i \subseteq \mathbb{P}^n$ defined as $Z(\{x_i\})$, and define $U_i := \mathbb{P}^n \smallsetminus H_i$. Then set

$$\begin{array}{rrcl} \varphi_i : & U_i & \longrightarrow & \mathbb{A}^n \\ & [a_0 : \dots : a_n] & \longmapsto & \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right) \end{array}.$$

The map φ_i is clearly bijective. Without loss of generality, assume $i = 0$. Let Y be a closed subset of U_0, and $\overline{Y}$ be its closure in $\mathbb{P}^n$. Then $\overline{Y} = Z(T)$ for some subset $T \subseteq S^h$ (where S^h denotes the subset of all homogeneous elements in S). Thus $\varphi_0(Y) = Z(T')$ where $T' \subseteq A = k[y_1, \ldots, y_n]$ is the subset

$$T' = \{f(1, y_1, \ldots, y_n) \mid f \in T\}.$$

Conversely, let F be a subset of A, then $\varphi_0^{-1}(Z(F)) = Z(F') \cap U_0$, where

$$F' := \{x_0^e g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \mid g \in F \text{ of degree } e\} \subseteq S = k[x_0, \ldots, x_n].$$

□

Before to define morphisms of varieties, we will briefly recall some constructions of sheaves, and in particular introduce the notion of sheaves of modules. For more general sheaf theory, we refer to ([3], I, 5) or ([8], II, 1).

Definition 3.4. Let $f : X \to Y$ be a continuous map between topological spaces, let $\mathcal{F}$ be a sheaf of abelian groups on X and $\mathcal{G}$ be a sheaf of abelian groups on Y. We define:

- the *direct image sheaf* on Y by $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$ for any open $V \subseteq Y$,
- the *inverse image sheaf* $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}$ for every open $U \subseteq X$, the limit taken on every open $V \subseteq Y$ containing $f(U)$.

In the case of the inclusion $i : Z \hookrightarrow X$ of a subspace Z of X (with the induced topology), we will often write $\mathcal{F}|_Z := i^{-1}\mathcal{F}$.

To define the next notion, recall that a *ringed space* $(X, \mathcal{O}_X)$ is the data of a topological space X and a sheaf of rings $\mathcal{O}_X$ over it.

Definition 3.5. Let $(X, \mathcal{O}_X)$ be a ringed space.

- A *sheaf of $\mathcal{O}_X$-modules* (or simply an $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ on X such that for every open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for any inclusion $V \subseteq U$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of $\mathcal{O}_X(U)$-modules.
- A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that the maps $\varphi(U)$ are morphism of $\mathcal{O}_X(U)$-modules.

Note that the kernel, cokernel, subsheaf, image, quotient, direct product, direct limit and inverse limit of $\mathcal{O}_X$-modules are $\mathcal{O}_X$-modules. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}|_U$ is a sheaf of $\mathcal{O}_X|_U$-modules.

Remark 3.6. A particular example of $\mathcal{O}_X$-module is the sheaf $\mathcal{O}_X^{\oplus n}$ called *free of rank n*. A $\mathcal{O}_X$-module $\mathcal{F}$ over X is said to be *locally free* if X can be covered by open subsets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module.

A locally free $\mathcal{O}_X$-module of rank 1 is called *invertible*. Invertible sheaves play an important role that we will discuss later in this text.

Another important example is the following: a *sheaf of ideals* $\mathcal{F}$ is an $\mathcal{O}_X$-module such that for all open $U \subseteq X$, $\mathcal{F}(U)$ is an ideal of $\mathcal{O}_X(U)$. In particular, $\mathcal{F}$ is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

Definition 3.7. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules. We define:

- the *sheaf Hom*, denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, to be the sheaf of $\mathcal{O}_X$-modules $U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ where the latter denotes the $\mathcal{O}_X(U)$-module of morphism of $\mathcal{O}_X|_U$-modules between $\mathcal{F}|_U$ and $\mathcal{G}|_U$;

- the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Now, let's look back to our construction of direct and inverse image of sheaves.

Definition 3.8. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, that is a couple $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then we define the $\mathcal{O}_X$-module $f^*\mathcal{G}$, called *inverse image of* $\mathcal{G}$ as:

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Given such an $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then the morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a structure of $\mathcal{O}_Y$-module on $f_*\mathcal{F}$.

Definition 3.9. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$, the map induced on stalks $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local morphism of rings, *i.e.* the preimage by $f^\#_x$ of the maximal ideal of $\mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{Y,f(x)}$.

In this definition, the induced map on stalks is defined as follows: the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces maps on stalks $f^\#_{f(x)} : \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)}$, but $(f_*\mathcal{O}_X)_{f(x)} = \mathcal{O}_{X,x}$ by definition of f_*.

In order to define morphisms of varieties, we will endow any variety with a structure of locally ringed space.

Let k be an algebraically closed field.

Definition 3.10. Let V be an affine variety in $\mathbb{A}^n_k$, let U be an open subset of V. A *regular function* on U is a map $f : U \to k$ such that for every point $x \in U$, there exists an open neighborhood $W \subseteq U$ containing x and polynomials $g, h \in A := k[x_1, \dots, x_n]$ such that $h \neq 0$ on W and $f = g/h$ on W.

One can check that a regular function is continuous if we identify k with $\mathbb{A}^1_k$ with the Zariski topology.

There is a similar definition for projective varieties.

Definition 3.11. Let P be an affine variety in $\mathbb{P}^n_k$, let U be an open subset of P. A *regular function* on U is a map $f : U \to k$ such that for every point $x \in U$, there exists an open neighborhood $W \subseteq U$ containing x and homogeneous polynomials $g, h \in S := k[x_0, \dots, x_n]$ of same degree such that $h \neq 0$ on W and $f = g/h$ on W.

Such g and h do not define functions on $\mathbb{P}^n_k$ but their quotient is well defined.

For now on, we will call variety any (quasi-)affine variety or (quasi-)projective variety.

Definition 3.12. Let X be a variety on k. For each open $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions on U. We define the *sheaf of regular functions* of X, denoted $\mathcal{O}_X$, to be the sheaf $U \mapsto \mathcal{O}(U)$ with obvious restriction maps.

The fact that $\mathcal{O}_X$ is a sheaf is not hard to check: a regular functions which is locally 0 is 0, and a functions which is locally regular is regular. Now we can consider a variety X as a ringed space $(X, \mathcal{O}_X)$.

Remark 3.13. The homeomorphisms defined in Propositon 3.3 induce morphisms of ringed spaces $(U_i, \mathcal{O}_{U_i}) \to (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$, *i.e.* we have the map $- \circ \varphi_i : \mathcal{O}_{\mathbb{A}^n} \to \varphi_{i*}\mathcal{O}_{U_i}$ which is a sheaf morphism.

Let's give some properties of $\mathcal{O}_X$.

Proposition 3.14. *Let X be an affine variety in $\mathbb{A}^n$, and define $A(X) = k[x_1, \dots, x_n]/I(Y)$. Then:*

1. *for each $x \in X$, $\mathcal{O}_{X,x} \simeq A(X)_{\mathfrak{m}_x}$ where $\mathfrak{m}_x$ is the (maximal) ideal of functions vanishing at x,*

2. *$\mathcal{O}_X(X) \simeq A(X)$.*

Proof.

1. For any $x \in X$ there is a natural morphism $A(X)_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$ which sends a quotient f/h, seen as a function defined locally near x, to its germ in the stalk $\mathcal{O}_{X,x}$. This map is clearly injective since the quotient does not depend on the choice of a neighborhood of x and it is surjective by definition of $\mathcal{O}_X$.

2. Notice that $\mathcal{O}_X(X) \subseteq \bigcap_{\mathfrak{m}_x} A(X)_{\mathfrak{m}_x}$ by definition of regular functions, where the $A(X)_{\mathfrak{m}_x}$ are seen as subrings of the quotient field of $A(X)$. Now, notice that the maximal ideals of $A(X)$ are exactly the $\mathfrak{m}_x$ for $x \in X$, thus we get

$$A(X) \subseteq O_X(X) \subseteq \bigcap_x A(X)_{\mathfrak{m}_x}.$$

To conclude, recall that any integral domain is equal to the intersection of its localizations at all maximal ideals.

□

An important consequence of this result is the fact that every stalk $\mathcal{O}_{X,x}$ is a local ring. Indeed, we proved earlier (cf. Remark 3.13) that any variety X can be covered by open subsets U_i such that $(U_i, \mathcal{O}_{U_i})$ is isomorphic (as a locally ringed space) to an affine variety $(V_i, \mathcal{O}_{V_i})$. We can now define the notion of morphism of variety.

Definition 3.15. Let X, Y be two varieties over an algebraically closed field k. A *morphism* from X to Y is a continuous map $\varphi : X \to Y$ such that the precomposition by φ induces a morphism of locally ringed space $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

In other words, a morphism between varieties is a continuous map which sends regular functions defined on an open $V \subseteq Y$ to regular functions defined on $f^{-1}(V) \subseteq X$. For instance, the homeomorphism $\varphi_i : U_i \to \mathbb{A}^n$ defined in Proposition 3.3 is an isomorphism of varieties. In particular, any variety can be covered by affine varieties (in the sense that the inclusion maps are homeomorphism on their images).

3.2 Schemes

Let A be a ring, let $\operatorname{Spec} A$ be the set of all primes ideal of A. For any ideal $\mathfrak{a}$ of A, let $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ be the set of all prime ideals which contain $\mathfrak{a}$.

Lemma 3.16. *We can define a topology on* $\operatorname{Spec} A$*, called the Zariski topology, by taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets of* $\operatorname{Spec} A$*. In particular,* $\operatorname{Spec} A = V(\{0\})$ *and $\emptyset = V(A)$.*

Proof. It's enough to show that $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$. See ([4], 2). □

We want to give a structure of ringed space to $\operatorname{Spec} A$.

Definition 3.17. We define a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$, called *structure sheaf of* $\operatorname{Spec} A$, as follows. For any open $U \subseteq \operatorname{Spec} A$, define $\mathcal{O}_{\operatorname{Spec} A}(U)$ as the ring of functions $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ which verify:

1. for all $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in A_{\mathfrak{p}}$,

2. for every $\mathfrak{p} \in U$, there is an open neighborhood V of $\mathfrak{p}$ and elements $a, f \in A$ such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$.

One can check that it is indeed a sheaf. Given a ring A, we call the ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ the *spectrum* of A. Notice that the open subsets $D(f)$, $f \in A$, defined as the complement of $V((f))$ form an open basis for the topology of $\operatorname{Spec} A$.

Proposition 3.18. *Let A be a ring and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be its spectrum. For any $\mathfrak{p} \in \operatorname{Spec} A$, we have $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$.*

Proof. Denote $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}$ to simplify the notation. Define the homomorphism $\varphi : \mathcal{O}_{\mathfrak{p}} \to A_{\mathfrak{p}}$ which sends any local section in a neighborhood of $\mathfrak{p}$ to $s(\mathfrak{p}) \in A_{\mathfrak{p}}$.

- φ is surjective: if $\mathfrak{p} \in D(f)$ then for any $a/f \in A_{\mathfrak{p}}$, $a, f \in A$ and $f \notin \mathfrak{p}$, the local section given by the image of a/f in the local rings over $D(f)$ takes the value a/f over $\mathfrak{p}$.
- φ is injective: assume $s(\mathfrak{p}) = t(\mathfrak{p})$. We can assume by taking a small enough neighborhood U of $\mathfrak{p}$ that $s = a/f$ and $t = b/g$ on U with $a, b, f, g \in A$ and $f, g \notin \mathfrak{p}$. Thus we have $a/f = b/g$ in $A_{\mathfrak{p}}$, so there is an $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$. Thus over $D(f) \cap D(g) \cap D(h)$, we have $s = t$, *i.e.* the local sections s and t coincide on an open neighborhood of $\mathfrak{p}$, so they are equal in $\mathcal{O}_{\mathfrak{p}}$.

$\square$

Remark 3.19. To be more precise, we have that for any element $f \in A$, the ring $\mathcal{O}_{\operatorname{Spec}(A)}(D(f))$ is isomorphic to the ring A_f. In particular, $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \simeq A$. In fact, it is possible to define $\mathcal{O}_X$ by setting $\mathcal{O}_X(D(f)) := A_f$ (see [4], 2).

Proposition 3.20. *If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces*

$$(f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

where $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a continuous map and $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$ is a morphism of sheaves.

Proof. We define $f : \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. It's easy to see that $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$ for any ideal $\mathfrak{a}$ of A, thus f is continuous. Now, for any prime ideal $\mathfrak{p} \subseteq B$ notice that an element in $A \setminus \varphi^{-1}(\mathfrak{p})$ is sent to a unit through the composition $A \to B \to B_{\mathfrak{p}}$, so by the universal property of the localization we obtain a homomorphism $\varphi_{\mathfrak{p}} : A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varphi_{\mathfrak{p}}"'] & B_{\mathfrak{p}} \\ A_{\varphi^{-1}(\mathfrak{p})} \arrow[ur] & \end{tikzcd}$$

commutes. Moreover $\varphi_{\mathfrak{p}}$ is local.

Now, for any open $V \subseteq \operatorname{Spec} A$ we want to define a morphism $f^{\#} : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(U))$. To do so, consider a section $\sigma \in \mathcal{O}_{\operatorname{Spec} A}(U)$, *i.e.* a map $\sigma : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$. Then $f^{\#}(U)(\sigma)$ is defined via the composition

$$f^{-1}(U) \xrightarrow{f} U \cap \operatorname{Im}(f) \xrightarrow{\sigma} \coprod A_{\varphi^{-1}(\mathfrak{q})} \xrightarrow{\amalg \varphi_{\mathfrak{q}}} \coprod B_{\mathfrak{q}}.$$

Finally, we identify the morphism $f^{\#}_{\mathfrak{p}}$ with the map $\varphi_{\mathfrak{p}}$ which is a local ring homomorphism, so $(f, f^{\#})$ is a morphism of locally ringed spaces. $\square$

Remark 3.21. In fact, the converse holds: any morphism of locally ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ arises from a homomorphism of rings $\varphi : A \to B$ ([8], II, 2.3).

Definition 3.22. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of a ring.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Example 3.23.
- If k is a field, $\operatorname{Spec} k$ is an affine scheme consisting in a singleton (viewed as a topological space) and the (constant) sheaf k.
- If k is an algebraically closed field, we define $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \dots, x_n]$, called the *affine space* over k. Hilbert's Nullstellensatz tells us that the closed points (*i.e.* the maximal ideals) are in one-to-one correspondance with the space k^n. Moreover, this correspondance yields an isomorphism between the set of closed point (with the induced topology) and the variety $\mathbb{A}^n_k$.

An affine scheme is always quasi-compact, but this is not true for general schemes. Usually, we will work with X being a *noetherian scheme*, that is a quasi-compact scheme such that X admits a covering of affine subsets $\operatorname{Spec} A_i$ with A_i's noetherian rings. It turns out that this is equivalent to the property that for *every* open affine subsets $\operatorname{Spec} A$, A is a noetherian ring. See ([8], II, 3.2).

Let S be a graded ring. Set $S_+ = \bigoplus_{d>0} S_d$, called *irrelevant* ideal of S. Let $\operatorname{Proj} S$ be the set of all homogeneous prime ideals $\mathfrak{p}$ of S which do not contain the irrelevant ideal S_+. For any homogeneous ideal $\mathfrak{a}$ of S, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S | \mathfrak{a} \subseteq \mathfrak{p}\}$.

Definition 3.24. We define a topology on $\operatorname{Proj} S$ by taking the closed subsets to be the subsets of the form $V(\mathfrak{a})$.

Recall that given a homogeneous ideal $\mathfrak{p} \in \operatorname{Proj} S$, the ring $S_{(\mathfrak{p})}$ is defined as the set of elements of degree 0 in the localized ring $T^{-1}S$, where T is the multiplicative system of all homogeneous elements of S which are not in $\mathfrak{p}$. We define a sheaf of rings $\mathcal{O}_{\operatorname{Proj} S}$ as follows. For any open $U \subseteq \operatorname{Proj} S$, define $\mathcal{O}_{\operatorname{Proj} S}(U)$ as the ring of functions $s : U \to \coprod_{\mathfrak{p} \in U} A_{(\mathfrak{p})}$ which verify:

1. for all $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$,
2. for every $\mathfrak{p} \in U$, there is an open neighborhood V of $\mathfrak{p}$ and homogeneous elements $a, f \in S$ of same degree such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$.

Proposition 3.25.
1. *For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.*
2. *For any homogeneous $f \in S_+$, let $D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S | f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in $\operatorname{Proj} S$, and these opens cover $\operatorname{Proj} S$.*
3. *For any homogeneous $f \in S_+$, we have*
$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}) \simeq \operatorname{Spec} S_{(f)},$$
where $S_{(f)}$ is the subring of elements of degree 0 in the localized ring S_f.

Proof. The proof of the first statement is similar to the proof of Proposition 3.18. The second statement is obvious since $D_+(f) = \operatorname{Proj} S \smallsetminus V((f))$, and that elements of $\operatorname{Proj} S$ are homogeneous prime ideals which do not contain the whole S_+.

To prove the last statement, we construct an isomorphism of locally ringed spaces $(\varphi, \varphi^{\#})$ from $D_+(f)$ to $\operatorname{Spec} S_{(f)}$. First, for any homogeneous ideal $\mathfrak{a}$ of S define $\varphi(\mathfrak{a}) := (\mathfrak{a}S_f) \cap S_{(f)}$. It's easy to verify that $\varphi(\mathfrak{p}) \in \operatorname{Spec} S_{(f)}$ whenever $\mathfrak{p} \in D_+(f)$ and that φ is bijective when restricted to $D_+(f)$. Moreover, $\mathfrak{a} \subseteq \mathfrak{p}$ if and only if $\varphi(\mathfrak{a}) \subseteq \varphi(\mathfrak{p})$. Hence φ is an homeomorphism.

To define $\varphi^{\#}$, consider an element $\sigma \in \mathcal{O}_{\operatorname{Spec} S_{(f)}}(U)$. Then we define $\varphi^{\#}(U)(\sigma)$ via the composition

$$\varphi^{-1}(U) \xrightarrow[\simeq]{\varphi} U \xrightarrow{\sigma} \coprod (S_{(f)})_{\varphi(\mathfrak{p})} \xrightarrow[\simeq]{} \coprod S_{\mathfrak{p}}$$

where the last map is given by the natural isomorphism $S_{(\mathfrak{p})} \simeq (S_{(f)})_{\varphi(\mathfrak{p})}$ for any homogeneous prime ideal $\mathfrak{p}$ which does not contain f. It is clear, since φ is an homeomorphism, that $\varphi^{\#}$ is an isomorphism of locally ringed spaces. □

Example 3.26.
- Let A be a ring and $A[x_0, \ldots, x_n]$ be the ring of polynomials in $n+1$ variables over A endowed with its natural graduation. We define $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \ldots, x_n]$ called *the projective n-space over A.*
- If $A = k$ is an algebraically closed field, then $\mathbb{P}^n_k$ is a scheme whose subset of closed point is isomorphic to the variety $\mathbb{P}^n_k$. Indeed, closed points are in one-to-one correspondance with homogeneous ideals which are maximal within $\operatorname{Proj} k[x_0, \ldots, x_n]$. Such ideals are of the form
$$< a_i x_j - a_j x_i \mid i, j = 0, \ldots, n > .$$
Notice $\mathbb{P}^n_k$ is covered by the distinguished open subsets $D_+(x_p)$ for $p = 0, \ldots, n$. Fix one of them (say $p = 0$), then on $D_+(x_0)$ you have the isomorphism
$$\begin{array}{rccc} \psi : & D_+(x_0) & \longrightarrow & \operatorname{Spec} k[x_0, \ldots, x_n]_{((x_0))} \\ & < a_i x_j - a_j x_i > & \longmapsto & < a_i \frac{x_j}{x_0} - a_j \frac{x_i}{x_0} > \end{array} .$$
We have the identification $\operatorname{Spec} k[x_0, \ldots, x_n]_{((x_0))} \simeq \operatorname{Spec} k[y_1, \ldots, y_n]$ by defining $y_j = \frac{x_j}{x_0}$, and then $< a_i \frac{x_j}{x_0} - a_j \frac{x_i}{x_0} >=< a_i y_j - a_j y_i >$.
In the case $n = 2$, every element in $D_+(x_0)$ is of the form $< a_0 x_1 - a_1 x_0 >$ with $a_0 \neq 0$, then its image by ψ is $< a_0 y_1 - a_1 >=< y_1 - \frac{a_1}{a_0} >\in \operatorname{Spec} k[y_1]$. We see that the change of coordinates on $D_+(x_0) \cap D_+(x_1)$ is exactly $[1 : \frac{a_1}{a_0}] \to [\frac{a_0}{a_1} : 1]$.

Definition 3.27. Let S be a fixed scheme. A *scheme over S* is a scheme X together with a morphism of schemes $X \to S$. A morphism $X \to Y$ between schemes over S (also called S-morphism) is a morphism of schemes such that the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

The next proposition shows how to relate the notions of scheme and variety.

Proposition 3.28. *Let k be an algebraically closed field. There is a natural fully faithful functor*

$$t : \mathbf{Var}/k \to \mathbf{Sch}/k$$

between the category of varieties over k to the category of schemes over k (that is, scheme over $\operatorname{Spec} k$*).*

Proof. Let V be a variety. We define the *topological space* $t(V)$ as the set of all non-empty irreducible closed subsets of X. Then, covering V by affine varieties V_i, we show that $t(V_i)$ is a scheme isomorphic to $(\operatorname{Spec} \mathcal{O}_{V_i}(V_i), \mathcal{O}_{V_i})$. See ([8], II, 2.6) for a complete proof. □

Remark 3.29. In fact, the image of this functor is exactly the set of quasi-projective integral schemes over k. The image of the set of projective varieties is the set of projective integral schemes. Hence, we can define an *abstract variety* to be an integral separated scheme of finite type over an algebraically closed field k. For more details see ([8], II, 4.10).

4 Coherent sheaves

4.1 (Quasi)-coherent sheaves

Definition 4.1. Let A be a ring and M be an A-module. We define the sheaf associated to M on $\operatorname{Spec} A$, denoted $\widetilde{M}$, as follows. For any open $U \subseteq \operatorname{Spec} A$, define $\widetilde{M}(U)$ as the ring of functions $s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ which verify:

1. for all $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$,
2. for every $\mathfrak{p} \in U$, there is an open neighborhood V of $\mathfrak{p}$ and elements $m \in M, f \in A$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$ in $M_{\mathfrak{q}}$.

Remark 4.2. The construction of $\widetilde{M}$ is very similar to the construction of $\mathcal{O}_{\operatorname{Spec} A}$. By the same type of argument as before, one can show that:

- $\widetilde{M}$ is an $\mathcal{O}_{\operatorname{Spec} A}$-module,
- for each $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $(\widetilde{M})_{\mathfrak{p}}$ is isomorphic to the localized module $M_{\mathfrak{p}}$,
- for any $f \in A$, the A_f-module $\widetilde{M}(D(f))$ is isomorphic to the localized module M_f, and in particular $\widetilde{M}(\operatorname{Spec} A) = M$.

Proposition 4.3. *Let A be a ring, $X = \operatorname{Spec} A$. Then for all A-modules M and N there is an isomorphism*

$$\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In particular, the functor $M \mapsto \widetilde{M}$ from the category of A-modules to the category of $\mathcal{O}_X$-modules is fully faithful.

Proof. A morphism between sheaves induces a morphism between the modules of global sections. We want to define an inverse $\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \to \operatorname{Hom}_A(M, N)$. Let $\varphi : M \to N$ be a homomorphism of A-modules. Then it induces a homomorphism $\varphi_f : M_f \to N_f$ of A_f-modules for all $f \in A$. Moreover, if $f, g \in A$ are such that $D(f) \subseteq D(g)$ then we have the commutative diagram

$$\begin{tikzcd} M_g \arrow[r, "\varphi_g"] \arrow[d] & N_g \arrow[d] \\ M_f \arrow[r, "\varphi_f"'] & N_f. \end{tikzcd}$$

Thus we obtain a well defined morphism of $\mathcal{O}_X$-modules $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$. □

Proposition 4.4. *Let A be a ring and let $X = \operatorname{Spec} A$. Then :*

1. *the functor $M \mapsto \widetilde{M}$ is exact,*
2. *if M and N are two A-modules, then $(\widetilde{M \otimes_A N}) = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,*
3. *If $(M_i)_{i \in I}$ is a family of A-modules, then $(\widetilde{\oplus M_i}) \simeq \oplus \widetilde{M_i}$.*

Proof. See ([4], 7.14). □

Definition 4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for all $x \in X$ there exists an open neighborhood U of x and an exact sequence (see Definition 1.7) of $\mathcal{O}_X|_U$-modules of the form

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0,$$

where I and J are arbitrary index sets depending on x.

Proposition 4.6. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the following assertions are equivalent.*

1. *For every open affine subset $U = \operatorname{Spec} A$ of X there exists an A-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$.*

2. *There exists an open affine covering $(U_i)_i$ of X, $U_i = \operatorname{Spec} A_i$, and for each i an A_i-module M_i such that $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$ for all i.*

3. *The $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent.*

4. *For every open affine subset $U = \operatorname{Spec} A$ of X and every $f \in A$ the homomorphism*

$$\mathcal{F}(U)_f \to \mathcal{F}(D(f))$$

is an isomorphism.

Proof. See ([4], 7.19). □

Corollary 4.7. *Let A be a ring and $X = \operatorname{Spec} A$. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category A-**Mod** of modules over the ring A and the category $\mathbf{Qcoh}(X)$ of quasi-coherent sheaves on X.*

The inverse functor is given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, where we denote $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. The same result is true for *coherent sheaves* (see below) if you consider the category of *finitely generated* A-modules.

Definition 4.8. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it satisfies the following properties :

1. $\mathcal{F}$ is of *finite type*, *i.e.* for all $x \in X$ there exists an open neighborhood U of x and a surjective morphism $\psi_U : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$.

2. For any such (U, ψ_U), the kernel of ψ_U is of finite type.

Proposition 4.9. *Let X be a noetherian scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the following assertions are equivalent.*

1. *$\mathcal{F}$ is coherent.*

2. *$\mathcal{F}$ is of finite presentation, i.e. for all $x \in X$ there exists a open neighborhood U of x and an exact sequence*
$$\mathcal{O}_X^n|_U \to \mathcal{O}_X^m|_U \to \mathcal{F}|_U \to 0.$$

3. *$\mathcal{F}$ is of finite type and quasi-coherent.*

Proof. It's clear that $1 \Rightarrow 2 \Rightarrow 3$. To prove $3 \Rightarrow 1$, since coherence is a local property, we may assume that $X = \operatorname{Spec} A$ for some noetherian ring A. By assumptions, there exists a morphism $\varphi : \mathcal{O}_X^n \to \mathcal{F}$. Since $\mathcal{F}$ is quasi-coherent, there exists a A-module M such that $\mathcal{F} \simeq \widetilde{M}$, and moreover M is finitely generated since $M \mapsto \widetilde{M}$ is an exact equivalence of category. But since A is noetherian, any finitely generated A-module is finitely presented. Using the equivalence again, we obtain that $\ker(\varphi)$ is of finite type. □

Remark 4.10. A quasi-coherent sheaf $\mathcal{F}$ on a noetherian scheme X is an $\mathcal{O}_X$-module which can locally be written $\mathcal{F}|_U = \widetilde{M}$ for some A-module M with A noetherian, and $\mathcal{F}$ is coherent if we can choose these modules to be finitely generated.

Proposition 4.11. *Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes and let $\varphi : A \to B$ be the corresponding ring homomorphism. Then:*

1. *For any B-module N, we have $f_*(\widetilde{N}) \simeq \widetilde{{}_AN}$, where ${}_AN$ is N considered as an A-module.*

2. *For any A-module M, we have $f^*(\widetilde{M}) \simeq (\widetilde{B \otimes_A M})$.*

Proof. To prove 1, notice that for all $g \in A$ we have $f^{-1}(D(g)) = D(\varphi(g))$ and therefore we obtain

$$f_*(\widetilde{N})(D(g)) = \widetilde{N}(D(\varphi(g))) \simeq N_{\varphi(g)} \simeq ({}_AN)_g \simeq \widetilde{{}_AN}(D(g)).$$

These identifications are compatible with restriction maps for $D(g') \subseteq D(g)$ and functorial in N, thus it proves 1.

To see 2, first consider the general result:

Lemma 4.12. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then for any $\mathcal{O}_X$-module $\mathcal{F}$ and every $\mathcal{O}_Y$-module $\mathcal{G}$ there exists an isomorphism of $\mathcal{O}_Y(Y)$-modules*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

In otherwords, f_* and f^* are adjoints. It follows from the isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

we refer to ([4], Proposition 2.27) for more details.

Now assuming Lemma 4.12 and using Proposition 4.3, for any quasi-coherent $\mathcal{O}_{\mathrm{Spec}\,A}$-module $\mathcal{F}$ we obtain the functorial isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}\,B}}(f^*\widetilde{M}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}\,A}}(\widetilde{M}, f_*\mathcal{F}), \\
&= \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}\,A}}(\widetilde{M}, \widetilde{{}_A\mathcal{F}(X)}), \\
&= \mathrm{Hom}_A(M, {}_A\mathcal{F}(X)), \\
&= \mathrm{Hom}_B(B \otimes_A M, \mathcal{F}(X)), \\
&= \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}\,B}}(\widetilde{B \otimes_A M}, \mathcal{F}).
\end{aligned}$$

Since $f^*\mathcal{O}_{\mathrm{Spec}\,A} = \mathcal{O}_{\mathrm{Spec}\,B}$ and f^* is exact and commutes with direct sums, we know that $f^*\widetilde{M}$ is quasi-coherent. Thus, we can apply the Yoneda lemma to conclude that

$$f^*\widetilde{M} \simeq \widetilde{B \otimes_A M}.$$

□

Corollary 4.13. *Let X be a noetherian scheme.*

1. *If $f : \mathcal{F} \to \mathcal{G}$ is a homomorphism of (quasi)-coherent $\mathcal{O}_X$-modules, then $\ker f$, $\mathrm{coker}\, f$, $\mathrm{Im}\, f$ are (quasi)-coherent $\mathcal{O}_X$-modules.*

2. *If $\mathcal{F}$ and $\mathcal{G}$ are (quasi)-coherent $\mathcal{O}_X$-modules, then $\mathcal{F} \oplus \mathcal{G}$ is (quasi)-coherent.*

3. *If $\mathcal{F}$ and $\mathcal{G}$ are (quasi)-coherent $\mathcal{O}_X$-modules, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is (quasi)-coherent.*

In particular, we see that the category of quasi-coherent (resp. coherent) sheaves $\mathbf{Qcoh}(X)$ (resp. $\mathbf{Coh}(X)$) is abelian.

Proof. The statements 1 and 2 are local, so they follow from Corollary 4.7. To see 3, it suffices to show that for any open affine subset $U = \mathrm{Spec}\,A \subseteq X$, we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U \simeq \mathcal{F}|_U \otimes_{\mathcal{O}_X|_U} \mathcal{G}|_U.$$

This isomorphism follows from the definition of the tensor product of $\mathcal{O}_X$-modules (via sheafification) and the fact that the isomorphisms

$$\widetilde{M}(D(f)) \otimes_{\widetilde{A}(D(f))} \widetilde{N}(D(f)) \simeq M_f \otimes_{A_f} N_f \simeq (M \otimes_A N)_f \simeq \widetilde{M \otimes_A N}(D(f))$$

are functorial and compatible with restrictions $D(g) \subseteq D(f)$. $\square$

Proposition 4.14. *Let $f : X \to Y$ be a morphism between two noetherian schemes.*

1. *If $\mathcal{G}$ is a (quasi)-coherent sheaf on Y, then $f^*\mathcal{G}$ is (quasi)-coherent on X.*

2. *If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent on Y.*

Proof.

1. Since the statement is local, we may assume that Y is affine. Moreover, since $(f^*\mathcal{G})|_U \simeq f^*(\mathcal{G}|_{f^{-1}(U)})$, we can also assume that X is affine. Then it follows from the local case (Proposition 4.11).

2. See ([4], Proposition 10.10) or ([8], Proposition 5.8).

$\square$

Remark 4.15. Be careful here: the direct image of a coherent sheaf is not coherent in general. However, it is the case when the morphism f have some nice properties, *e.g.* finite or projective.

4.2 Support and closed immersions

Consider a ringed space $(X, \mathcal{O}_X)$.

Definition 4.16. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}$$

is called the *support* of $\mathcal{F}$.

Note that the support of $\mathcal{F}$ is not closed in general. We have the following result:

Proposition 4.17. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point and let $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$, be sections over some open neighborhood U of x such that the germs $(s_i)_x$ generate the stalk $\mathcal{F}_x$. Then there exists an open neighborhood $V \subseteq U$ such that the $s_i|_V$ generates $\mathcal{F}|_V$.*

Proof. Let $U' \subseteq U$ be an open neighborhood of x such that $\mathcal{F}|_{U'}$ is generated by sections $t_j \in \mathcal{F}(U')$, $j = 1, \ldots, m$. Then there exist sections a_{ij} of $\mathcal{O}_X$ over an open neighborhood $U'' \subseteq U'$ of x such that $(t_j)|_x = \sum_i (a_{ij})_x (s_i)_x$ for all j. Therefore there exists an open neighborhood $V \subseteq X$ of x such that $t_j|_V = \sum_i a_{ij}|_V \cdot s_i|_V$. In particular, $(s_i)_y$ generate $\mathcal{F}_y$ for all $y \in V$. $\square$

Corollary 4.18. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then* $\mathrm{Supp}\,\mathcal{F}$ *is closed in X.*

Proof. Note that $\mathrm{Supp}\,\mathcal{F}$ is the complement of the subset $X_0 := \{x \in X \mid \mathcal{F}_x = 0\}$. But for any $x \in X_0$, the 0-section generates $\mathcal{F}_x$. Then it generates $\mathcal{F}|_V$ for some open neighborhood $V \subseteq X$ of x, and thus X_0 is open. $\square$

Remark 4.19. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with closed support $Z \subseteq X$. Then for all $U, V \subseteq X$ containing Z, we have $\mathcal{F}(U) = \mathcal{F}(V)$. Indeed, $\mathcal{F}(U)$ identifies with the set of maps $\sigma : U \to \bigcup\limits_{x \in U} \mathcal{F}_x$ which verifies $\sigma(x) \in \mathcal{F}_x$ and which image can locally be lifted to sections of $\mathcal{F}$. But since $\mathcal{F}_x = 0$ for $x \notin Z$, maps in $\mathcal{F}(U)$ and $\mathcal{F}(V)$ identify. In particular, if $Z = \{x\}$ is a singleton, we can think $\mathcal{F}$ as an $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

Definition 4.20. Let Y, X be two schemes. A *closed immersion* is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of Y onto a closed subset of X, and furthermore the induced morphism $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. We say that Y is *a closed subscheme* of X.

Example 4.21.
- If A is a ring and $\mathfrak{a}$ is an ideal of A, then $Y := \operatorname{Spec}(A/\mathfrak{a})$ is a closed subscheme of $X := \operatorname{Spec} A$. Indeed, the image of Y in X is $V(\mathfrak{a})$ and the morphism on sheaves is surjective since it is surjective on stalks which are localization of A and $A/\mathfrak{a}$ respectively.

- Let X be a scheme and x be a closed point in X. Since $\mathcal{O}_{X,x}$ is local, the quotient $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}$ is a field, called the *residual field* at x. Then $(\operatorname{Spec}\kappa(x), \kappa(x))$ is a closed subscheme of X: a closed immersion is given by $\operatorname{Spec}\kappa(x) \ni * \mapsto x \in X$ and the composition $\mathcal{O}_X \to \mathcal{O}_{X,x} \to \kappa(x)$. This structure on $\{x\}$ is called *reduced induced closed subscheme structure.*

- If X is a scheme over k and x is a closed point of X endowed with the reduced closed subscheme structure over k (that is, over $\operatorname{Spec} k$), then the composition $\operatorname{Spec}\kappa(x) \to X \to \operatorname{Spec} k$ gives a homomorphism of fields $k \to \kappa(x)$. In fact, this extension is finite by Hilbert's Nullstellensatz (see [4], 3.33); thus if k is algebraically closed every closed point has residual field k.

Definition 4.22. Let Y be a closed subscheme of a scheme X and let $i : Y \to X$ be the inclusion morphism. We define the *ideal sheaf* of Y, denoted $\mathcal{I}_Y$, as the kernel of the morphism $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Proposition 4.23. *Let X be a noetherian scheme. Then there is a one-to-one correspondance between the set of closed subschemes of X and the set of quasi-coherent sheaves of ideals of $\mathcal{O}_X$ given by $Y \mapsto \mathcal{I}_Y$.*

Proof. If Y is a closed subscheme of X, then Y is noetherian so the sheaf $i_*\mathcal{O}_Y$ is quasi-coherent on X. Thus $\mathcal{I}_Y$ is also quasi-coherent.

Now, given a quasi-coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$, consider the couple $(Y, \mathcal{O}_Y)$ with $Y := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ a subspace of X and $\mathcal{O}_Y := i_Y^{-1}(\mathcal{O}_X/\mathcal{I})$ a sheaf of rings on Y, where $i_Y : Y \to X$ is the inclusion map. Since $\mathcal{O}_X/\mathcal{I}$ is an $\mathcal{O}_X$-module of finite type, its support is closed by Corollary 4.18. Because the properties of being a scheme and of being quasi-coherent can both be checked locally, we may assume that $X = \operatorname{Spec} A$ is an affine scheme. Now $\mathcal{I}$ is quasi-coherent if and only if there exists an ideal $\mathfrak{a}$ of A such that $\mathcal{I} \simeq \widetilde{\mathfrak{a}}$. Indeed, by Corollay 4.7, $\mathcal{I}$ is of the form $\widetilde{M}$ for some A-module M, and the injective morphism $\mathcal{I} \to \mathcal{O}_X$ induces an injective morphism $M \to A$, *i.e.* M is an ideal of A. But then $Y = V(\mathfrak{a})$ and $\mathcal{O}_Y = \widetilde{A/\mathfrak{a}}$, and hence $Y = \operatorname{Spec}(A/\mathfrak{a})$ (see Example 4.21). □

This proof implies immediately the following:

Corollary 4.24. *If $X = \operatorname{Spec} A$ is an affine scheme, there is a one-to-one correspondance between ideals of A and closed subschemes of X. In particular, every closed subscheme of an affine scheme is affine.*

Now we give a brief description of "tangent vectors" on a scheme.

Definition 4.25. Let X be a scheme over a field k, and let $x \in X$ be a point. Denote by $\mathfrak{m}_x$ the maximal ideal of the stalk $\mathcal{O}_{X,x}$ and by $k(x)$ the residual field $\mathcal{O}_{X,x}/\mathfrak{m}_x$. We define the *Zariski cotangent space* of X in x to be the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. The $k(x)$-dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the *(Zariski) tangent space* of X in x.

Proposition 4.26. *Let X be a scheme over a field k. Then the data of a rational point $x \in X$ (i.e. such that $k(x) = k$) endowed with a tangent direction $\mu : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$ is equivalent to the data of a k-subscheme $Z_x \hookrightarrow X$ concentrated in $\{x\}$ of length two (i.e. $\Gamma(Z_x, \mathcal{O}_{Z_x})$ is a k-vector space of dimension 2).*

Proof. Suppose we have such a subscheme Z_x. We have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[rr, "u"] & & \mathcal{O}_{Z_x}, \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where we identify the sheaf $\mathcal{O}_{Z_x}$ with a local ring $(\mathcal{O}_{Z_x}, \mathfrak{m}_{Z_x})$. Now $\mathfrak{m}_{Z_x}^p$ is a k-vector subspace of $\mathcal{O}_{Z_x}$ for all integer $p > 0$ and since $\cap_p \mathfrak{m}^p = 0$ (Krull intersection theorem, see [2], III, §3, 2) and $\dim_k \mathcal{O}_{Z_x} = 2$ we have $\mathfrak{m}_{Z_x}^2 = 0$ and $\mathfrak{m}_{Z_x} = k.m$ for some $m \in \mathfrak{m}_{Z_x}$.

Hence one define a morphism $\mu : \mathfrak{m}_x \to k$ by composing $u|_{\mathfrak{m}_x} : \mathfrak{m}_x \to \mathfrak{m}_{Z_x}$ (recall that u is a local ring morphism) with the projection $\mathfrak{m}_{Z_x} \to k$ given by $\lambda \cdot m \mapsto m$, $\lambda \in k$. Since $\mathfrak{m}_{Z_x}^2 = 0$, the morphism μ factors through $\mathfrak{m}_x/\mathfrak{m}_x^2$ and we obtain a tangent vector

$$\mu : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k.$$

Conversely, suppose we are given a point $x \in X$ and a morphism $\mu : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. We want to extend such a map to a morphism of local rings $\mathcal{O}_{X,x} \to k[T]/T^2$ (so that we obtain the subscheme Z_x defined as the skysraper sheaf $k[T]/T^2$ concentrated in x). We have the structural k-algebra morphism $k \overset{i}{\hookrightarrow} \mathcal{O}_{X,x}$ and the (k-algebra) projection $\mathcal{O}_{X,x} \overset{\pi}{\twoheadrightarrow} \mathcal{O}_{X,x}/\mathfrak{m}_x \simeq k$.

Choosing $\nu \in k$ to be the representant of $\pi \circ i(\nu)$, we obtain that $\pi \circ i = \mathrm{Id}$. For all $a \in \mathcal{O}_{X,x}$ there is a unique decomposition $a = i \circ \pi(a) + m_a$ for some $m_a \in \mathcal{O}_{X,x}$. Moreover, $\pi(a) = \pi \circ i \circ \pi(a) + \pi(m_a) = \pi(a) + \pi(m_a)$ and thus $m_a \in \mathfrak{m}_x$. Thus we define the morphism

$$\begin{array}{rccl} u : & \mathcal{O}_{X,x} & \longrightarrow & k[T]/T^2 \\ & a & \longmapsto & \pi \circ i(a) + \mu(m_a)T \end{array}$$

and combined with the canonical injection $k \ni \nu \to \nu + 0.T \in k[T]/T^2$ we obtain a k-subscheme $Z_x := k[T]/T^2$ of length two. □

5 Projective schemes

5.1 Ampleness

Let S be a graded ring and $X = \operatorname{Proj} S$. For any $n \in \mathbb{Z}$, we define $S(n)$ as the ring S with a shifted graduation: for all $k \in \mathbb{Z}$, $s \in S$ has degree k if and only if $s \in S(n)$ has degree $k - n$.

Definition 5.1. For any $n \in \mathbb{Z}$, let $\mathcal{O}_X(n)$ be the $\mathcal{O}_X$-module $\widetilde{S(n)}$, and for any $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{F}(n)$ be the $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. We call $\mathcal{O}_X(1)$ the *twisting sheaf of Serre.*

Proposition 5.2. *Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra.*

1. *The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf of X (that is locally free of constant rank 1).*
2. *For any graded S-module M, $\widetilde{M}(n) \simeq \widetilde{M(n)}$. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$.*

Proof. See ([8], II, Proposition 5.12) □

Notice that for any homogeneous element $f \in S_+$, as $S_{(f)}$ is the group of elements of degree 0 in S_f, $S(n)_{(f)}$ is the group of elements of degree n in S_f.

Definition 5.3. Let X be a scheme, and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there is a family of global sections $\{s_i\}_{i\in I}$, $s_i \in \Gamma(X, \mathcal{F})$ such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. Equivalently, it means that we have a surjective map

$$\Gamma(X, \mathcal{F}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}.$$

In particular, $\mathcal{F}$ is generated by global sections if and only if $\mathcal{F}$ can be written as the quotient of a free sheaf.

If $X = \mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$, then $\Gamma(X, \mathcal{O}_X(n))$ is the space of homogeneous polynomials of degree n and then $\mathcal{O}_X(n)$ is generated by global sections.

In the following, let X be a scheme over a ring A. To simplify the notations, we will denote $\mathcal{O}(n) := \mathcal{O}_{\mathbb{P}^n_A}(n)$.

Definition 5.4. An invertible sheaf $\mathcal{L}$ on X is said to be *very ample* relative to $\operatorname{Spec} A$ (or relative to A) if there exists an *immersion* $i : X \to \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$ for some n, such that $i^*(\mathcal{O}(1)) \simeq \mathcal{L}$. We say that a morphism $X \to Z$ is an immersion if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.

Theorem 5.5. *1. If $\varphi : X \to \mathbb{P}^n_A$ is an A-morphism, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X which is generated by the global sections $s_i = \varphi^*(x_i)$, $i = 0, \ldots, n$.*

2. Conversely, if $\mathcal{L}$ is an invertible sheaf on X, if $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \simeq \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.

Proof. 1. It's clear that $\varphi^*(\mathcal{O}(1))$ is invertible. Indeed, for any affine subset $U \subseteq \mathbb{P}^n_A$ such that $\mathcal{O}(1)|_U \simeq \mathcal{O}_{\mathbb{P}^n_A}|_U$, consider an affine cover $\varphi^{-1}(U) = \bigcup V_i$. Then for each i the restricted morphism $\varphi_i : V_i \to U$ is a morphism of affine schemes, then $\varphi^*(\mathcal{O}(1))|_{V_i} \simeq \varphi^*(\mathcal{O}_U) \simeq \mathcal{O}_{V_i}$. Moreover, since $\varphi^*(\mathcal{O}(1))_x = \mathcal{O}(1)_{\varphi(x)} \otimes_{\mathcal{O}_{\mathbb{P}^n_A,x}} \mathcal{O}_{X,\varphi(x)}$, we see that the global sections $\varphi^*(x_i)$, $i = 0, \ldots, n$, generate $\varphi^*(\mathcal{O}(1))$.

2. For each $i = 0, \ldots, n$ let $X_i := \{x \in X \mid (s_i)_x \notin \mathfrak{m}_x \mathcal{L}_x\}$, where $\mathcal{L}_x$ is identified with $\mathcal{O}_{X,x}$. First, notice that X_i is open: in any open affine subset $U = \operatorname{Spec} B \subseteq X$, we have an identification $s_i|_U = b \in B$. Then $X_i \cap U = D(b)$, in particular X_i is open in any open affine subset of X, thus it is open in X. Since the s_i's generate $\mathcal{L}$, the sets X_i must cover X.

Now consider the standard open subsets $U_i := \{x_i \neq 0\} \simeq \operatorname{Spec} A[y_0, \ldots, y_n]$ of $\mathbb{P}^n_A$, where $y_j = x_j/x_i$ and y_i is omitted. Define the ring morphism

$$\mu : A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$$

sending y_j to s_j/s_i and making it A-linear. Notice that by construction, the morphism $\mathcal{O}_{X_i} \to \mathcal{L}_{X_i}$ given by $1 \mapsto s_i = s_i|_{X_i}$ is an isomorphism since it is invertible on stalks (the inverse is given by $1 \to 1/s_i$). Hence we can identify s_j/s_i as an element of $\Gamma(X_i, \mathcal{O}_{X_i})$. The morphism μ induces a morphism of schemes (over A) $X_i \to U_i$ (see [8], II, Exercice 2.4), and these morphisms glue together so we obtain an A-morphism

$$\varphi : X \to \mathbb{P}^n_A.$$

We see by contruction that we have $\varphi^*(x_i) = s_i$ and thus $\varphi^*(\mathcal{O}(1)) \simeq \mathcal{L}$. Moreover, the uniqueness is quite clear since any A-morphism with these properties would be as constructed on each open X_i.

□

Theorem 5.6. *Let k be an algebraically closed field, let X be a projective scheme over k and let $\varphi : X \to \mathbb{P}^n_k$ be a k-morphism corresponding to the invertible sheaf $\mathcal{L}$ and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ (see Theorem 5.5). Let $V \subseteq \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i's. Then φ is a closed immersion if and only if:*

1. *Elements of V separate points, i.e. for any two distinct closed points $x, y \in X$ there is an $s \in V$ such that $s \in \mathfrak{m}_x \mathcal{L}_x$ but $s \notin \mathfrak{m}_y \mathcal{L}_y$ or vice-versa.*

2. *Elements of V separate tangent vectors (or tangent directions), i.e. for each closed point $x \in X$ the set $\{s \in V | s_x \in \mathfrak{m}_x \mathcal{L}_x\}$ spans the k-vector space $\mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$.*

Proof. Assume $\varphi : X \hookrightarrow \mathbb{P}^n$ is a closed immersion. Then $\mathcal{L}$ identifies with $\mathcal{O}_X(1)$ and the vector space V is spanned by the images of $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$. Given two closed points $x, y \in X$, there exists a homogeneous polynomial f of degree 1 such that $f(x) = 0$ and $f(y) \neq 0$ (here we use that k is algebraically closed). Hence $f|_X$ is a global sections of $\mathcal{L}$ that satisfies condition 1: just identify, for any closed point $t \in X$, the stalk $\mathfrak{m}_t \mathcal{L}_t$ with the germs of regular functions of $\mathcal{L}$ that vanish at t.

For condition 2, assume for simplicity that $x = [1 : 0 : \dots : 0]$. Then x is in the distinguished open U_0 which identifies with $\mathbb{A}^1_k = \operatorname{Spec} k[y_1, \dots, y_n]$ with $y_i = x_i / x_0$. In U_0, x identifies with the point $(0, \dots, 0)$ and the sheaf $\mathcal{L}|_{U_0}$ becomes trivial (*i.e.* isomorphic to $\mathcal{O}_{\mathbb{A}^n_k}$) via the map

$$\frac{f}{g} \mapsto \frac{f(1, y_1, \dots, y_n)}{g(1, y_1, \dots, y_n)},$$

where f and g are homogeneous polynomials with $\deg f = \deg g + 1$. Hence, the stalk $\mathfrak{m}_x$ is given by germs of regular functions that are of the form f/g near 0, with $f(0) = 0$ and $g(0) \neq 0$. In particular, such an f is of the form

$$f = a_1 y_1 + \cdots + a_n y_n + \text{higher degree terms}$$

with $a_j \in k$. But the higher degree terms are in $\mathfrak{m}_x^2$, and thus $\mathfrak{m}_x / \mathfrak{m}_x^2$ is generated by the global sections $y_1, \dots, y_n$.

For the converse, see ([8], II, 7.3). The idea is the following: we want the sections $s_0, \dots, s_n$ to be our coordinates in $\mathbb{P}^n_k$. Separating points ensures that φ is injective (the fact that φ is an homeomorphism onto a closed subset of $\mathbb{P}^n_k$ can be verified using the projectivity of X). For the surjectivity of the morphism of sheaves $\mathcal{O}_{\mathbb{P}^n_k} \to \varphi_* \mathcal{O}_X$ (*i.e.* the surjectivity of the induced morphism on stalks $\mathcal{O}_{\mathbb{P}^n_k, x} \to \mathcal{O}_{X,x}$), we want to apply the following result of commutative algebra:

Lemma 5.7. *Let $f : A \to B$ be a local morphism of local noetherian rings such that*

1. *$A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism*

2. *$\mathfrak{m}_A \to \mathfrak{m}_B / \mathfrak{m}_B^2$ is surjective, and*

3. *B is a finitely generated A-module.*

Then f is surjective.

(A proof of this lemma is also given in [8], II, 7.3).

The morphism $\mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{X,x}$ verify condition 1 since both residual fields are k, and condition 3 is ensured by the fact that $\varphi_* \mathcal{O}_X$ is coherent (Remark 4.15). Finally, separating points gives condition 2, hence φ is a closed immersion. □

Remark 5.8. Notice that V separates points if and only if the restriction map

$$V \to \Gamma(X, k(x) \oplus k(y))$$

is surjective for any two closed points $x, y \in X$. Similarly, V separates tangent vectors if and only if, for any subscheme Z_x of lenght two concentrated in $x \in X$, the restriction

$$V \to \Gamma(X, \mathcal{O}_{Z_x})$$

is surjective (see Proposition 4.26).

Definition 5.9. An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be *ample* if for every coherent sheaf $\mathcal{F}$ on X there is an integer $n_0 > 0$ such that for any $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections.

Our use of ampleness lies in the following theorem.

Theorem 5.10. *Let X be a variety over a field k and $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over* $\operatorname{Spec} k$ *for some $m > 0$.*

Proof. As the proof is long and technical, we refer to the litterature ([8], II, Theorem 7.6 or [4], Theorem 13.59). □

Notice that the hypothesis on X can be weakened (X must be a scheme of finite type over a noetherian ring A), but we will try to avoid properties of scheme in this text as our goal is to study projective varieties.

5.2 Cohomology and projective space

In this part, we work with a ringed space $(X, \mathcal{O}_X)$. We will give brief recalls on cohomology of sheaves, which is directly related to derived functors (see Definition 2.4). We will use the following famous result, and we refer to the literature ([8], III, 2) for a proof.

Proposition 5.11. *The categories* $\mathbf{Ab}$ *of abelian groups, the category* $\mathbf{Ab}_X$ *of sheaves of abelian groups on X and the category* $\mathbf{Sh}_{\mathcal{O}_X}(X)$ *of sheaves of $\mathcal{O}_X$-modules have enough injectives.*

Definition 5.12. For all $n \in \mathbb{Z}$, we define the *cohomology functors* $H^n(X, \ )$ to be the right derived functors $R^n\Gamma(X, \ )$ of the left exact functor $\Gamma : \mathbf{Ab}_X \to \mathbf{Ab}$ sending a sheaf $\mathcal{F}$ on X to its global sections $\Gamma(X, \mathcal{F})$.

Recall that given a sheaf $\mathcal{F}$ and an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, we have $H^n(X, \mathcal{F}) = H^n(\Gamma(X, \mathcal{I}^\bullet))$. In particular, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ (see Remark 2.4).

Remark 5.13. The cohomology groups $H^n(X, \mathcal{F})$ can also be computed by other resolutions $\mathcal{F} \to \mathcal{G}^\bullet$: we just need the resolution to be Γ-acyclic, *i.e.* $H^n(X, \mathcal{G}^\bullet) = 0$ for all $n > 0$. For instance, *flasque* sheaves (that is sheaves with all restriction maps being surjective) are Γ-acyclic.

We can define other usefull derived functors:

Definition 5.14. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For all $n \in \mathbb{Z}$, we define:

- the functors $\operatorname{Ext}^n(\mathcal{F}, \ )$ to be the right derived functors of the functor $\operatorname{Hom}(\mathcal{F}, \ )$,
- the functors $\mathcal{E}xt^n(\mathcal{F}, \ )$ to be the right derived functors of the functor $\mathcal{H}om(\mathcal{F}, \ )$.

Proposition 5.15. *For any $\mathcal{O}_X$-module $\mathcal{G}$, we have:*

1. $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$,
2. $\mathcal{E}xt^n(\mathcal{O}_X, \mathcal{G}) = 0$ *for $n > 0$,*
3. $\operatorname{Ext}^n(\mathcal{O}_X, \mathcal{G}) \simeq H^n(X, \mathcal{G})$ *for all $n \geq 0$.*

Proof. First note that $\operatorname{Hom}(\mathcal{O}_X, \mathcal{G}) = \Gamma(X, \mathcal{G})$ since for all open subset $U \subseteq X$ any morphism $\mathcal{O}_X(U) \to \mathcal{G}(U)$ is totally defined by the image of $1_{\mathcal{O}_X(U)}$ and thus can be lifted in a unique way to a morphism between global sections. In particular, it means that the functors $\operatorname{Hom}(\mathcal{O}_X, \ )$ and $\Gamma(X, \ )$ are equal and then their derived functors are equal. It proves 3.

Now, $\mathcal{H}om(\mathcal{O}_X, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{O}_X|_U, \mathcal{G}|_U) = \mathcal{G}(U)$ for any open subset $U \subseteq X$. Thus we have $\mathcal{H}om(\mathcal{O}_X, \ ) = \mathrm{Id}$. It proves 1 and 2. □

Proposition 5.16. *Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{L}$ as the sheaf $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then for any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we have*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L} \otimes \mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}).$$

Proof. First, let's prove that $\mathcal{L}^\vee \otimes \mathcal{G} \simeq \mathcal{H}om(\mathcal{L}, \mathcal{G})$. Indeed, for any open subset $U \subseteq X$, define

$$\begin{array}{rrcl} \mu : & \mathcal{L}^\vee(U) \otimes \mathcal{G}(U) & \longrightarrow & \operatorname{Hom}(\mathcal{L}|_U, \mathcal{G}|_U) \\ & l \otimes g & \longmapsto & (\mathcal{L}(V) \ni \sigma \mapsto l(\sigma) \cdot g|_V \in \mathcal{G}(V)) \end{array}$$

for any open subset $V \subseteq U$. It is clearly an isomorphism on stalks, so the map μ^+ induced by the sheafification is also an isomorphism.

Now, define the map

$$\varphi : \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L} \otimes \mathcal{F}, \mathcal{G}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om(\mathcal{L}, \mathcal{G}))$$

as follows. For any $\psi : \mathcal{L} \otimes \mathcal{F} \to \mathcal{G}$, set $\varphi(\psi)_U(s)_V(t) := \psi_V(t \otimes s|_V)$, where $V \subseteq U \subseteq X$ are open subsets, $s \in \mathcal{F}(U)$, $t \in \mathcal{L}(V)$ and $t \otimes s|_V$ is the image of the element in the presheaf $\mathcal{L} \otimes \mathcal{F}$ via the sheafification. Now $\varphi(\psi) = 0$ if and only if ψ_V is the zero map for every V. Thus φ is injective. Finally, given a morphism $\theta : \mathcal{F} \to \mathcal{H}om(\mathcal{L}, \mathcal{G})$, define for any $U \subseteq X$ the map $\psi_U : \mathcal{L}(U) \otimes \mathcal{F}(U) \to \mathcal{G}(U)$ as $\psi_U(l \otimes f) = \theta(f)(l)$ for any $l \in \mathcal{L}(U)$, $f \in \mathcal{F}(U)$. Then the associated map

$$\psi^+ : \mathcal{L} \otimes \mathcal{F} \to \mathcal{G}$$

verifies $\varphi(\psi^+) = \theta$, hence φ is an isomorphism. □

If $\mathcal{L}$ is a locally free sheaf of rank 1, it is easy to see that $\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X$. That's the reason why we say that $\mathcal{L}$ is an *invertible* sheaf.

Corollary 5.17. *Let $\mathcal{L}$ be a locally free sheaf of finite rank. Then for any $\mathcal{O}_X$-module $\mathcal{F}, \mathcal{G}$ we have*

$$\operatorname{Ext}^n(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \operatorname{Ext}^n(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}).$$

Proof. Notice that if $\mathcal{I}$ is injective, then $\mathcal{L} \otimes \mathcal{I}$ is also injective. Indeed, the functor $\mathcal{L}^\vee \otimes (\ )$ is exact and the functors $\operatorname{Hom}(\ \ , \mathcal{L} \otimes \mathcal{I})$ and $\operatorname{Hom}(\cdot \otimes \mathcal{L}^\vee, \mathcal{I})$ are isomorphic. In particular, it means that if $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{G}$, then $\mathcal{L}^\vee \otimes \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{L}^\vee \otimes \mathcal{G}$. Since the functors $\operatorname{Hom}(\mathcal{F} \otimes \mathcal{L}, \ \ )$ and $\operatorname{Hom}(\mathcal{F}, \mathcal{L}^\vee \otimes \cdot)$ are isomorphic, so are their derived functors. □

Remark 5.18. Similar results exist for the sheaves $\mathcal{E}xt^n(\mathcal{F}, \mathcal{G})$, but since the category $\mathbf{Sh}_{\mathcal{O}_X}(X)$ does not contain enough projectives in general, one has to use more general theory, e.g. δ-functors, see ([5], II, 2.2.1).

Theorem 5.19. *Let X be a projective variety over a field k and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then there exists an isomorphism of k-schemes*

$$X \simeq \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^d).$$

Proof. For more details, see ([4], 13.75).

By Theorem 5.10, up to consider tensor powers we can suppose that $X \overset{\varphi}{\hookrightarrow} \mathbb{P}^n$ and $\mathcal{O}_X(1) := \mathcal{L} = \varphi^*(\mathcal{O}(1))$. Let $\mathcal{I}_X$ be sheaf of ideals determined by X in $\mathbb{P}^n$. Recall that $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. Then we have

$$X = \operatorname{Proj} k[x_0, \ldots, x_n]/I,$$

where $I = \Gamma_*(\mathcal{I}_X) := \bigoplus_{k\in\mathbb{Z}} \Gamma(X, \mathcal{I}_X(k))$ (see [8], II, 5.16). Then define the morphism of graded algebras

$$\mu : k[x_0, \ldots, x_n] \longrightarrow \bigoplus_{k\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(k))$$

given by $P(x_0, \ldots, x_n) \mapsto P(s_0, \ldots, s_n)$ where the s_i's are the sections defining φ. It is clear that $I = \ker \mu$, so we just need to check surjectivity. To do so, consider the exact sequence

$$0 \to \mathcal{I}_X \otimes \mathcal{O}(k) \to \mathcal{O}(k) \to \mathcal{O}_X(k) \to 0.$$

It induces a long exact sequence in cohomology

$$\cdots \to H^0(X, \mathcal{O}(k)) \to H^0(X, \mathcal{O}_X(k)) \to H^1(X, \mathcal{O}(k) \otimes \mathcal{I}_X) \to \cdots .$$

But $\mathcal{O}(k) \otimes \mathcal{I}_X = \mathcal{I}_X(k)$ and by Serre vanishing theorem ([8], III, 5.2) we obtain, for $k \gg 0$, that $H^1(X, \mathcal{I}_X(k)) = 0$ and thus $H^0(X, \mathcal{O}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective. Hence μ is surjective. □

This result is fundamental as it will be a key step in the proof of the main theorem presented in this text (Theorem 7.9).

5.3 Serre duality

To begin with, we will give a definition of differential forms for algebraic varieties, but in order to avoid scheme-theoretic constructions we will do it in a local-to-global way. For deeper studies, see ([8], II, 8).

Definition 5.20.

- Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}^n_k$ be an affine variety over k defined by the polynomials $(f_1, \ldots, f_m)$. The set of *algebraic differential forms* on V, denoted Ω_V, is the $\Gamma(V, \mathcal{O}_V)$-module generated by the symbols $dx_1, \ldots, dx_n$ with relations $df_1, \ldots, df_m$, where $df_i = \sum \frac{\partial f_i}{\partial x_j}$.
- Let X be a variety over k. Consider a covering $X = \bigcup V_i$ by affine open subsets. We define the *sheaf of differential forms* Ω_X over X to be the sheaf given by glueing the sheaves $\widetilde{\Omega_{V_i}}$. The maps $d_i : \Gamma(V_i, \mathcal{O}_{V_i}) \to \Omega_{V_i}$ glue together to give a map $d : \mathcal{O}_X \to \Omega_X$.
- We define
$$\Omega^q_X := \Lambda^q \Omega_X.$$
Its elements are called *q-forms* or *differential forms of degree q*.

Recall that an affine variety $V \subseteq \mathbb{A}^n$ defined by polynomials $f_1, \ldots, f_m$ is *regular* (or *non-singular*) at a point $x \in V$ if the matrix $(\frac{\partial f_i}{\partial x_j})$ has rank $(n - \dim V)$. A variety X is *non-singular* at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is *regular*, that is $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$. In the affine case both definitions coincide.

A non-singular variety X is a variety which is non-singular at every point. We also say that X is smooth.

Theorem 5.21. *Let X be an irreducible non-singular variety over an algebraically closed field k. Then Ω_X is a locally free sheaf of rank* $\dim X$.

Proof. See ([8], II, 8.15). □

Definition 5.22. Let X be a non-singular variety over an algebraically closed field k. We define the *canonical sheaf* of X to be

$$\omega_X := \Lambda^n \Omega_X.$$

Behind this definition, there is the deep notion of *dualizing* sheaves. We refer to ([8], III, 7) for more details.

The canonical sheaf has the following (dualizing) property: there exists a *trace* morphism

$$t : H^n(X, \omega_X) \to k$$

such that for any coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$$

followed by t gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \simeq H^n(X, \mathcal{F})^*.$$

Theorem 5.23. *Let X be a non-singular projective variety of dimension n over an algebraically closed field k. Let ω_X be its canonical sheaf. Then, for all $i \geq 0$ and for all $\mathcal{F}$ coherent sheaf on X, there are natural functorial isomorphisms*

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^*$$

where the * stands for the dual vector space, such that θ^0 is the map given by the dualizing property discussed above.

Proof. See ([8], III, 7.6) for a complete proof.

The idea of the proof is the following. To construct the morphisms θ^i, one needs to prove that any coherent sheaf $\mathcal{F}$ can be written as a quotient of a sheaf of the form $\bigoplus_{j=1}^{n} \mathcal{O}_X(-q)$ for some $q \gg 0$, where $\mathcal{O}_X(1)$ is a very ample sheaf on X. Then, one proves that for any locally free sheaf $\mathcal{F}$ on X, we have

$$H^i(X, \mathcal{F}(-q)) = 0$$

for $i < n$ and q big enough. To conclude that the θ^i's are isomorphisms, one can use the theory of δ-functors (see [5], II, 2.2.1).

□

Remark 5.24. If $\mathcal{F}$ is locally free, we have

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \simeq \mathrm{Ext}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X) \simeq H^i(X, \mathcal{F}^\vee \otimes \omega_X)$$

and then Serre duality gives

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*.$$

Part III
Derived category of coherent sheaves

In this last part we will use all the notions and tools introduced in Part I and II. Our principal goal is the proof of the last result (Theorem 7.9) due to Bondal and Orlov.

6 Derived category and canonical bundle

Let X be a noetherian scheme. We study $\mathrm{D}^b(X)$, the bounded derived category of coherent sheaves on X. Note that if X is defined over a field k, then the derived category will be considered as a k-linear category.

To avoid any confusion with cohomology of sheaves, we will denote the n^{th} cohomology *sheaf* (though as an object of the abelian category $\mathbf{Coh}(X)$) of a complex of sheaves $\mathcal{F}^\bullet$ as $\mathcal{H}^n(\mathcal{F}^\bullet)$.

6.1 Basic structure

The category of coherent sheaves on a noetherian scheme does not contain enough injectives in general, but we have the following result.

Proposition 6.1. *On a noetherian scheme X, any quasi-coherent sheaf $\mathcal{F}$ admits a resolution*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

by quasi-coherent sheaves $\mathcal{I}^n$ which are injective as $\mathcal{O}_X$-modules.

Proof. As the proof is long and technical, we refer to the literature ([7], II, 7.18). □

In particular, we obtain that $\mathbf{Qcoh}(X)$ has enough injectives whenever X is at least noetherian. In particular, it permits us to use the spectral sequences defined in Proposition 2.21. Thus for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{Qcoh}(X)$ we have:

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{F}^\bullet, \mathcal{H}^q(\mathcal{G}^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{2}$$

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{3}$$

Proposition 6.2. *Let X be a noetherian scheme. Then the natural functor*

$$\mathrm{D}^b(X) \to \mathrm{D}^b(\mathbf{Qcoh}(X))$$

defines an equivalence between the derived category $\mathrm{D}^b(X)$ of X and the full triangulated subcategory $\mathrm{D}^b_{coh}(\mathbf{Qcoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.

Proof. Let $\mathcal{G}^\bullet$ be a bounded complex of quasi-coherent sheaves

$$\cdots \to 0 \to \mathcal{G}^n \to \cdots \to \mathcal{G}^m \to 0 \to \cdots$$

with coherent cohomology $\mathcal{H}^i$, $i = n, \dots, m$. Suppose that $\mathcal{G}^i$ is coherent for $i > j$. The construction of a new complex quasi-isomorphic to $\mathcal{G}^\bullet$ with $\mathcal{G}^j$ coherent relies on the following lemma:

Lemma 6.3. *If $\mathcal{G} \to \mathcal{F}$ is a surjective morphism of $\mathcal{O}_X$-modules from a quasi-coherent sheaf $\mathcal{G}$ onto a coherent sheaf $\mathcal{F}$ on a noetherian scheme X, then there exists a coherent subsheaf $\mathcal{G}' \subseteq \mathcal{G}$ such that the composition $\mathcal{G}' \subseteq \mathcal{G} \to \mathcal{F}$ is still surjective.*

This lemma is clear locally: for any surjection $M \to N$ of modules with N finitely generated, there exists a finitely generated submodule $M' \subset M$ such that the restriction $M' \to N$ is still surjective. The step to the global case is not trivial but can be found in the literature ([8], II, 5.15).

Now, apply this lemma to the surjections

$$d^j : \mathcal{G}^j \twoheadrightarrow \mathrm{Im}(d^j) \quad \text{and} \quad \ker(d^j) \twoheadrightarrow \mathcal{H}^j$$

which yield subsheaves $\mathcal{G}_1^j \subseteq \mathcal{G}^j$ and $\mathcal{G}_2^j \subseteq \ker(d^j)$. Now define $\widetilde{\mathcal{G}^j} \subseteq \mathcal{G}^j$ the coherent subsheaf generated by $\mathcal{G}_1^j$ and $\mathcal{G}_2^j$, and define $\widetilde{\mathcal{G}^{j-1}}$ as the pre-image of $\widetilde{\mathcal{G}^j}$ under the morphism $\mathcal{G}^{j-1} \to \mathcal{G}^j$. We get the injective morphism of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}^{j-2} \arrow[r] \arrow[d, "="] & \widetilde{\mathcal{G}^{j-1}} \arrow[r, "\widetilde{d^{j-1}}"] \arrow[d, hook, "i_{j-1}"] & \widetilde{\mathcal{G}^j} \arrow[r] \arrow[d, hook, "i_j"] & \mathcal{G}^{j+1} \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & \mathcal{G}^{j-2} \arrow[r] & \mathcal{G}^{j-1} \arrow[r] & \mathcal{G}^j \arrow[r] & \mathcal{G}^{j+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Notice that i_j induces an isomorphism in cohomology by construction of $\mathcal{G}_2^j$, and the $(j+1)^{th}$ cohomology group of the first row is still $\mathcal{H}^{j+1}$ by construction of $\mathcal{G}_1^j$. Finally, $\widetilde{\mathcal{G}^{j-1}}$ is constructed so that $\widetilde{d^{j-1}}$ is well defined. Thus this morphism of complexes is a quasi-isomorphism and $\widetilde{\mathcal{G}^j}$ is coherent. $\square$

Remark 6.4. If X is a projective variety over a field k, for any coherent sheaf $\mathcal{F}$ the groups $H^n(X, \mathcal{F})$ are finite-dimensional (Serre theorem, [14], III, §3, 66 or [8], III, 5.2).

This result can be used to show (by induction) that for any two coherent sheaves $\mathcal{F}, \mathcal{G}$ the groups $\mathrm{Ext}^n(\mathcal{F}, \mathcal{G})$ are also finite-dimensional for all $n \in \mathbb{Z}$. Indeed, the case $n = 0$ comes from the identity $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$. The case $\mathcal{F} = \bigoplus \mathcal{L}_j$ with $\mathcal{L}_j$ locally free sheaves of finite rank comes from the equality

$$\begin{aligned}
\mathrm{Ext}^n(\bigoplus \mathcal{L}_j, \mathcal{G}) &\simeq \bigoplus \mathrm{Ext}^n(\mathcal{L}_j, \mathcal{G}), \\
&\simeq \bigoplus \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{L}^\vee \otimes \mathcal{G}) \quad \text{by Corollary 5.17}, \\
&\simeq \bigoplus H^n(X, \mathcal{L}^\vee \otimes \mathcal{G}) \quad \text{by Proposition 5.15}.
\end{aligned}$$

Finally, one conclude using that any coherent sheaf can be placed in an exact sequence

$$0 \to \mathcal{K} \to \bigoplus \mathcal{L}_j \to \mathcal{F} \to 0$$

with $\mathcal{L}_j$ locally free ([8], II, 5.18). Applying $\mathrm{Hom}(\ ,\mathcal{G})$ we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^n(\mathcal{K}, \mathcal{G}) \to \mathrm{Ext}^{n+1}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^{n+1}(\bigoplus \mathcal{L}_j, \mathcal{G}) \cdots$$

(see Remark 2.5). Now since the first term is finite-dimensional by induction and the last term is finite-dimensional by the previous case, the middle one is also finite-dimensional.

Eventually, using both spectral sequences (2) and (3), we obtain that $\mathrm{Ext}^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is finite-dimensional.

Definition 6.5. The *support* of a complex $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ is the union of the supports of all its cohomology sheaves, *i.e.* it is the closed subset (as finite union of closed subsets):

$$\mathrm{Supp}(\mathcal{F}^\bullet) := \bigcup \mathrm{Supp}(\mathcal{H}^n(\mathcal{F}^\bullet)).$$

Lemma 6.6. *Suppose that* $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ *and* $\mathrm{Supp}(\mathcal{F}^\bullet) = Z_1 \coprod Z_2$*, where* $Z_1, Z_2 \subseteq X$ *are disjoint closed subsets. Then* $\mathcal{F}^\bullet \simeq \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$ *with* $\mathrm{Supp}(\mathcal{F}_j^\bullet) \subseteq Z_j$ *for* $j = 1, 2$.

Proof. We proceed by induction on the length of the complex. For a complex of length 1, the result is quite clear. Indeed, up to a shift we can assume that $\mathcal{F}^\bullet = \mathcal{F} \in \mathbf{Coh}(X)$. Then $\mathcal{F} \simeq \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\mathcal{F}_j := i_{j*}i_j^*\mathcal{F}$ for the closed immersion $i_j : Z_j \to X$, $j = 1, 2$. Then the morphism of sheaves

$$\begin{array}{rrcl} \mu : & \mathcal{F} & \longrightarrow & \mathcal{F}_1 \oplus \mathcal{F}_2 \\ & \mathcal{F}(U) \ni s & \longmapsto & (s|_{U\cap(X\setminus Z_2)}, s|_{U\cap(X\setminus Z_1)}) \in \mathcal{F}_1(U) \oplus \mathcal{F}_2(U) \end{array}$$

is clearly bijective on stalks.

Now let $\mathcal{F}^\bullet$ be a sheaf of length at least 2. Assume m is minimal with $\mathcal{H} := \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. Using the previous step, we have a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathrm{Supp}(\mathcal{H}_j) \subseteq Z_j$. Now define $\widetilde{\mathcal{F}}^\bullet$ as the complex

$$\cdots \to \mathcal{F}^{m-1} \to \mathcal{F}^m \to \mathrm{Im}\, d^m \to 0 \to \cdots$$

which is quasi-isomorphic to $\mathcal{H}$ up to a shift. Hence the roof $\mathcal{H}[-m] \leftarrow \widetilde{\mathcal{F}}^\bullet \to \mathcal{F}$ induces a natural arrow (in $\mathrm{D}^b(X)$) $\mathcal{H}[-m] \to \mathcal{F}^\bullet$ which can be completed into a distinguished triangle

$$\mathcal{H}[-m] \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}[1-m].$$

The long exact sequence in cohomology (see Proposition 1.46) shows that $\mathcal{H}^q(\mathcal{G}^\bullet) = \mathcal{H}^q(\mathcal{F}^\bullet)$ for $q > m$ and $\mathcal{H}^q(\mathcal{G}^\bullet) = 0$ for $q \leq m$. Thus, the induction hypothesis applies to $\mathcal{G}^\bullet$ and we may write $\mathcal{G}^\bullet = \mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet$ with $\mathrm{Supp}(\mathcal{H}^q(\mathcal{G}_j^\bullet)) \subseteq Z_j$ for all q. Now, consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}[p]) \Rightarrow \mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[p+q]).$$

In order to prove that $\mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[1-m]) = 0$, one uses the following lemma:

Lemma 6.7. *If $\mathcal{F}, \mathcal{G}$ are two $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ with disjoint supports, then for all $n \in \mathbb{Z}$ we have $\mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) = 0$.*

This lemma is quite clear: $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is 0 since it is 0 on every stalks. For $n \geq 1$, we have $\mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{G}[n])$ (see Proposition 2.9). But the latter is 0 since any roof $\mathcal{F} \leftarrow \mathcal{K}^\bullet \to \mathcal{G}$ is trivial. Indeed, the arrow $\mathcal{K}^\bullet \to \mathcal{F}$ is a quasi-isomorphism, so in particular $\mathrm{Supp}(\mathcal{K}^\bullet) = \mathrm{Supp}(\mathcal{F})$ and thus the morphism $\mathcal{K}^\bullet \to \mathcal{G}$ is 0.

Similarly, using again the lemma one finds that $\mathrm{Hom}(\mathcal{G}_2^\bullet, \mathcal{H}_1[1-m]) = 0$. To finish the proof, choose $\mathcal{F}_j$, $j = 1, 2$, to complete the arrows $\mathcal{G}_j^\bullet \to \mathcal{H}_j[1-m]$ to distinguished triangles

$$\mathcal{F}_j^\bullet \to \mathcal{G}_j^\bullet \to \mathcal{H}_j[1-m] \to \mathcal{F}_j^\bullet[1].$$

We obtain the diagram

$$\begin{tikzcd} \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \arrow[r] \arrow[d, dotted, "h"] & \mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet \arrow[r] \arrow[d, "\sim"] & \mathcal{H}_1^\bullet[1-m] \oplus \mathcal{H}_2^\bullet[1-m] \arrow[r] \arrow[d, "\sim"] & \mathcal{F}_1^\bullet[1] \oplus \mathcal{F}_2^\bullet[1] \arrow[d, dotted, "h[1]"] \\ \mathcal{F}^\bullet \arrow[r] & \mathcal{G}^\bullet \arrow[r] & \mathcal{H}[1-m] \arrow[r] & \mathcal{F}^\bullet \end{tikzcd}$$

where $h : \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \to \mathcal{F}$ is given by the axiom TR-3, and which is moreover an isomorphism by Lemma 1.14. Using the long exact sequence in cohomology one checks that $\mathcal{H}^m(\mathcal{F}_j^\bullet) \simeq \mathcal{H}_j$ and $\mathcal{H}^q(\mathcal{F}_j^\bullet) \simeq \mathcal{H}^q(\mathcal{G}_j^\bullet)$; in particular we have $\mathrm{Supp}(\mathcal{F}_j) \subseteq Z_j$ as required. $\square$

Theorem 6.8 (Serre duality)**.** *Let X be a smooth projective variety over a field k. Then*

$$S_X : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$$

which sends $\mathcal{F}^\bullet$ to $\mathcal{F}^\bullet \otimes \omega_X[n]$ is a Serre functor (see Definition 1.4), i.e. for any two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in \mathrm{D}^b(X)$ there exists a functorial isomorphism

$$\eta : \mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Ext}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^*$$

Proof. The derived version of Serre duality is based on the usual one (Theorem 5.23).

Recall that we have $\operatorname{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = H^i(R\operatorname{Hom}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet))$ (see Proposition 2.12). Up to replace $\mathcal{E}^\bullet$ by a complex of locally free sheaves and $\mathcal{F}^\bullet$ by a complex of injective sheaves, we have $R\operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Moreover,

$$\begin{aligned}
\operatorname{Hom}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) &= \bigoplus \operatorname{Hom}(\mathcal{E}^k, \mathcal{F}^{k+i}), \\
&\simeq \bigoplus \operatorname{Ext}^0(\mathcal{O}_X, (\mathcal{E}^k)^\vee \otimes \mathcal{F}^{k+i}), \\
&\simeq \operatorname{Ext}^n(\mathcal{F}^{k+i}, \mathcal{E}^k \otimes \omega_X)^*, \\
&\simeq \operatorname{Hom}(\mathcal{F}^{k+i}, \mathcal{E}^k \otimes \omega_X[n])^*, \\
&\simeq \operatorname{Hom}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^*,
\end{aligned}$$

and thus the desired isomorphism is obtained by replacing $\mathcal{E}^\bullet \otimes \omega_X$ by a complex of injective objects and taking cohomology. □

Corollary 6.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on a smooth projective variety X of dimension n. Then*

$$\operatorname{Ext}^n(\mathcal{F}, \mathcal{G}) = 0 \text{ for } i > n.$$

Proof. Simply notice that $\operatorname{Ext}^p(\mathcal{F}, \mathcal{G}) = 0$ for $p < 0$ and for all $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$. □

Corollary 6.10. *Let X be a smooth projective variety. Then for any two complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ one has $R\operatorname{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in \mathrm{D}^b(\mathbf{Ab})$.*

Proof. Simply apply Corollary 6.9 and the spectral sequences (2) and (3). □

Proposition 6.11. *Let X be a smooth projective variety. Then the objects of the form $k(x)$ with $x \in X$ a closed point span the derived category $\mathrm{D}^b(X)$ (see Definition 1.19).*

Proof. Using Serre duality, we only need to prove that for any non-trivial $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ there exist a closed point $x \in X$ and an integer $i \in \mathbb{Z}$ such that

$$\operatorname{Hom}(\mathcal{F}^\bullet, k(x)[i]) \neq 0.$$

Consider the spectral sequence (2)

$$E_2^{p,q} = \operatorname{Hom}(\mathcal{H}^{-q}, k(x)[p]) \Rightarrow \operatorname{Hom}(\mathcal{F}^\bullet, k(x)[p+q])$$

where $\mathcal{H}^q := \mathcal{H}^q(\mathcal{F}^\bullet)$. Let m be maximal such that $\mathcal{H}^m \neq 0$. Then all differentials with source $E_r^{0,-m}$ are trivial, and since negative Ext groups between coherent sheaves are trivial all differentials with target $E_r^{0,-m}$ are also trivial. Thus $E_2^{0,-m} = E_\infty^{0,-m}$.

Now let $x \in \operatorname{Supp}(\mathcal{H}^m)$. Then

$$E_\infty^{0,m} = \operatorname{Hom}(\mathcal{H}^m, k(x)) \neq 0,$$

hence $\operatorname{Hom}(\mathcal{F}^\bullet, k(x)[-m]) \neq 0$. □

6.2 Derived functors

In this section we will briefly recall results on derived functors that will be used in the next part. For more precise studies, see ([9], 3.3).

Let X be a smooth projective variety over a field k. Note that $\mathbf{Coh}(X)$ has not enough injectives neither projectives in general. However, we still have the following result.

Proposition 6.12. *The functors*

$$\Gamma : \mathbf{Coh}(X) \to \mathbf{Vect}_f(k),$$

$$\mathrm{Hom}(\mathcal{F}, \quad) : \mathbf{Coh}(X) \to \mathbf{Vect}_f(k),$$
$$\mathcal{H}om(\mathcal{F}, \quad) : \mathbf{Coh}(X) \to \mathbf{Coh}(X),$$
$$\mathcal{F} \otimes - : \mathbf{Coh}(X) \to \mathbf{Coh}(X),$$

admit derived functors between the ***bounded*** *derived categories, where* $\mathbf{Vect}_f(k)$ *is the category of* k*-vector spaces of finite dimension and* $\mathcal{F}$ *is a coherent sheaf on* X*.*

We will not give a proof, but just say few words about it.

- To construct $R\Gamma : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathbf{Vect}_f(k))$, we use that $\dim_k H^i(X, \mathcal{F}) < \infty$ and $H^i(X, \mathcal{F}) = 0$ for $i > \dim(X)$ (see [8], III, 2.7). Then, we construct the derived functor using the compositions $\mathrm{D}^b(X) \to \mathrm{D}^b(\mathbf{Qcoh}(X)) \to \mathrm{D}^b(\mathbf{Vect}_f(k))$ with Proposition 6.2.

- Global Hom functor $\mathrm{Hom}^\bullet(\ ,\ )$ has already been treated in Proposition 2.12. Notice that boundedness is ensured by Serre duality (see Corollary 6.10).

- The derived functor $R\mathcal{H}om(\mathcal{F}, \quad)$ is defined using the fully faithful functor $\mathrm{D}^b(X) \to \mathrm{D}^b(\mathbf{Qcoh}(X))$ (one check that if $\mathcal{F}, \mathcal{G}$ are coherent, so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$). We can generalize the constructions as follows.

  The exact functor $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \quad) : \mathrm{K}^b(\mathbf{Coh}(X)) \to \mathrm{K}^b(\mathbf{Coh}(X))$ is defined by

  $$\mathcal{H}om^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \prod \mathcal{H}om(\mathcal{F}^p, \mathcal{G}^{p+i}), \quad d = d_{\mathcal{G}} - (-1)^i d_{\mathcal{F}}.$$

  We proceed as before: we check that $\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is acyclic whenever $\mathcal{F}^\bullet$ or $\mathcal{G}^\bullet$ is acyclic, then we use that $\mathbf{Qcoh}(X)$ has enough injectives.

  In fact, we also have a bifunctor

  $$R\mathcal{H}om(\ ,\ ) : \mathrm{D}^b(X)^{op} \times \mathrm{D}^b(X) \to \mathrm{D}^b(X)$$

  To see it, it suffices to show that $R\mathcal{H}om(\mathcal{F}^\bullet, \ ) = \mathcal{H}om(\mathcal{F}^\bullet, \ )$ if $\mathcal{F}^\bullet$ is a complex of locally free sheaves (*i.e.* complexes of locally free sheaves form an adapted subcategory for this functor, see Proposition 2.6). Since X is regular, any complex $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ is isomorphic to a complex $\mathcal{G}^\bullet$ of locally free sheaves. This last claim follows from the fact that on a smooth scheme any coherent sheaf $\mathcal{F}$ admits a locally free resolution of *finite* length (see [8], III, ex.6.9).

- The functor $\mathcal{F} \otimes -$ is a *right* exact functor. One checks that the class of locally free sheaves in $\mathbf{Coh}(X)$ is adapted to this functor (see Corollary 2.7), and thus we obtain a derived functor

  $$\mathcal{F} \otimes^L - : \mathrm{D}^b(X) \to \mathrm{D}^b(X),$$

  where boundedness is ensured by smoothness of X.

  We have a more general situation. Let $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ be a bounded complex. We define the exact functor $\mathcal{F}^\bullet \otimes - : \mathrm{K}^b(X) \to \mathrm{K}^b(X)$ as

  $$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i := \bigoplus_{p+q=i} \mathcal{F}^p \otimes \mathcal{G}^q, \quad d = d_{\mathcal{F}} \otimes 1 + (-1)^i 1 \otimes d_{\mathcal{G}}.$$

  Then it suffices to check that the subcategory of complexes of locally free sheaves is adapted to this functor and we obtain the derived functor

  $$\mathcal{F}^\bullet \otimes^L - : \mathrm{D}^b(X) \to \mathrm{D}^b(X).$$

In fact, we also have a bifunctor

$$- \otimes^L - : \mathrm{D}^b(X) \times \mathrm{D}^b(X) \to \mathrm{D}^b(X),$$

because $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is acyclic whenever $\mathcal{F}^\bullet$ is acyclic and $\mathcal{G}^\bullet$ is a complex of locally free sheaves (in other words, the functor $- \otimes -$ needs not to be derived in the first factor). Notice that have the isomorphisms

$$\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet \simeq \mathcal{G}^\bullet \otimes^L \mathcal{F}^\bullet,$$

$$\mathcal{E}^\bullet \otimes^L (\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) \simeq (\mathcal{E}^\bullet \otimes^L \mathcal{F}^\bullet) \otimes^L \mathcal{G}^\bullet.$$

7 Main results

In this section we will present some results, mostly due to Bondal and Orlov ([1]). All derived functors will be written with their non-derived notation (*e.g* $\otimes$ will stand for $\otimes^L$) as we will always work in the bounded derived categories of coherent sheaves on a smooth projective variety.

Proposition 7.1. *Let X and Y be smooth projective varieties over a field k. If there exists an exact equivalence*

$$\mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$$

of their derived categories, then

$$\dim(X) = \dim(Y).$$

Moreover, their canonical bundles ω_X and ω_Y are of the same order.

Recall that the order of an invertible sheaf $\mathcal{L}$, if it exists, is the smallest positive integer $m \in \mathbb{Z}$ such that $\mathcal{L}^m$ is trivial. If there is no such integer, we say that $\mathcal{L}$ is of *infinite* order.

Proof. Since both varieties are smooth projective, their derived categories $\mathrm{D}^b(X)$ and $\mathrm{D}^b(Y)$ come with natural Serre functors S_X and S_Y (see 6.8). By Proposition 1.5, the equivalence F commutes with S_X and S_Y.

Fix a closed point $x \in X$. Then $k(x) \simeq k(x) \otimes \omega_X \simeq S_X(k(x))[-\dim X]$ and hence

$$\begin{aligned} F(k(x)) &\simeq F(S_X(k(x))[-\dim X]), \\ &\simeq F(S_X(k(x)))[-\dim X], \text{ since } F \text{ is exact}, \\ &\simeq S_Y(F(k(x)))[-\dim X], \\ &\simeq F(k(x)) \otimes \omega_Y[\dim Y - \dim X]. \end{aligned}$$

Since F is an equivalence, $F(k(x))$ is non-trivial. Let m be maximal (resp. minimal) with $\mathcal{H}^m(F(k(x))) \neq 0$. Notice that $\mathcal{H}^k(\mathcal{F}^\bullet \otimes \mathcal{L}) \simeq \mathcal{H}^k(\mathcal{F}^\bullet) \otimes \mathcal{L}$ for any complex $\mathcal{F}^\bullet$, integer k and line bundle $\mathcal{L}$. This comes from the fact that line bundles become trivial on stalks. Hence we obtain:

$$\begin{aligned} 0 &\neq \mathcal{H}^m(F(k(x))), \\ &\simeq \mathcal{H}^{m+\dim Y - \dim X}(F(k(x)) \otimes \omega_Y, \end{aligned}$$

and hence $0 \neq \mathcal{H}^{m+\dim Y - \dim X}$ which contradicts the maximality (resp. minimality) of m if $\dim Y \neq \dim X$. Hence, we have $n := \dim X = \dim Y$.

To finish, assume $\omega_X^k \simeq \mathcal{O}_X$. Then $S_X^k[-kn] \simeq \mathrm{Id}$ and hence

$$F^{-1} \circ S_Y^k[-kn] \circ F \simeq S_X^k[-kn] \simeq \mathrm{Id}\,.$$

Therefore, $S_Y^k[-kn] \simeq \mathrm{Id}$ and thus $\omega_Y^k \simeq \mathcal{O}_Y$. Since we can perform the previous isomorphisms switching the roles of X and Y, we obtain that ω_X^k is trivial if and only if ω_Y^k is trivial. $\square$

In order to prove Theorem 7.9, we need to characterize the geometry of our variety intrinsically as objects in the derived category.

Definition 7.2. Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S. An object $P \in \mathcal{D}$ is called *point like of codimension d* if

1. $S(P) \simeq P[d]$,
2. $\mathrm{Hom}(P, P[n]) = 0$ for all $n < 0$,
3. $k(P) := \mathrm{Hom}(P, P)$, endowed with the composition, is a field.

An object P verifying 3 is called *simple.* Notice that P is simple if and only if every non-trivial endomorphism $P \to P$ is invertible.

Remark 7.3. For any closed point $x \in X$, where X is a smooth projective variety, the skyscraper sheaf $k(x)$ and all its shifts $k(x)[m]$, $m \in Z$, are point like objects of codimension n. Indeed, conditions 1 and 3 are obvious, and condition 2 also holds since $\mathrm{Hom}(\mathcal{F}, \mathcal{F}[p]) = \mathrm{Ext}^p(\mathcal{F}, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ and any $p < 0$.

Lemma 7.4. *Let X be a smooth projective variety and let $\mathcal{F}^\bullet$ be a simple object in $\mathrm{D}^b(X)$ with zero-dimensional support. If $\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet[n]) = 0$ for $n < 0$, then*

$$\mathcal{F}^\bullet \simeq k(x)[m]$$

for some closed point $x \in X$ and some integer m.

Proof. First, we prove that $\mathrm{Supp}(\mathcal{F}^\bullet)$ is concentrated in one point. If not, we would have a non-trivial decomposition $\mathcal{F}^\bullet \simeq \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$ (see Lemma 6.6). But the projection onto one of the two summands is not invertible.

Thus, we may assume that all cohomology sheaves $\mathcal{H}^n$ of $\mathcal{F}^\bullet$ have support concentrated in a closed point $x \in X$. Set

$$m_0 := \max\{n \mid \mathcal{H}^n \neq 0\} \text{ and } m_1 := \min\{n \mid \mathcal{H}^n \neq 0\}.$$

Recall the following fact of commutative algebra: if M is a finite A-module with $(A, \mathfrak{m})$ a Noetherian local ring, and $\mathrm{Supp}(M) = \{\mathfrak{m}\}$, then there exists a surjection $M \twoheadrightarrow A/\mathfrak{m}$ and an injection $A/\mathfrak{m} \hookrightarrow M$.

Indeed, since A is Noetherian and M is finite there exists a prime filtration $0 \subset M_0 \subset \cdots \subset M_n \subset M$ such that $M_{i+1}/M_i \simeq A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i \in \mathrm{Supp}(M)$ (see [11], Chapter 3). Thus we have $\mathfrak{p}_i = \mathfrak{m}$ for all i. In particular, we have the projection $M \twoheadrightarrow M/M_n \simeq A/\mathfrak{m}$. On the other hand, since $M \neq 0$, there is a $\mathfrak{p} \in \mathrm{Supp}\, M$ such that $\mathfrak{p} = \mathrm{Ann}(\{m\})$ for some $m \in M$. Thus, $\mathfrak{m}$ is the kernel of the map

$$\begin{array}{rccc} f: & A & \longrightarrow & M \\ & a & \longmapsto & am \end{array},$$

and thus there is an injective morphism $f : A/\mathfrak{m} \hookrightarrow M$.

Now, consider an affine neighborhood $U = \mathrm{Spec}\, A$ of x. We have $k(x) \simeq A/\mathfrak{m}_x$. The coherent sheaves $\mathcal{H}^{m_0}$ and $\mathcal{H}^{m_1}$ are given on U by A-modules M_0 and M_1, and since $\{x\} = \mathrm{Supp}(\mathcal{H}^{m_0}) = \mathrm{Supp}(\mathcal{H}^{m_1})$ we have

$$\mathcal{H}_x^{m_0} \simeq (M_0)_{\mathfrak{m}_x} \neq 0 \text{ and } \mathcal{H}_x^{m_1} \simeq (M_1)_{\mathfrak{m}_x} \neq 0,$$

and $\mathrm{Supp}(M_0) = \mathrm{Supp}(M_1) = \{\mathfrak{m}\}$. Then by the fact stated above there exist $(M_0)_{\mathfrak{m}_x} \twoheadrightarrow A/\mathfrak{m}_x$ and $A/\mathfrak{m}_x \hookrightarrow (M_1)_{\mathfrak{m}_x}$. Using Corollary 4.7, the composition yields a non-trivial morphism

$\mathcal{H}^{m_0}_x \to \mathcal{H}^{m_1}_x$ which extends to a non-trivial morphism $\mathcal{H}^{m_0} \to \mathcal{H}^{m_1}$ since both sheaves are supported in x.

Now, in the same vein as in the proof of Lemma 6.6, consider the complexes

$$\mathcal{F}^\bullet_i : \cdots \to \mathcal{F}^{m_i-1} \to \ker(d^{m_i}) \to 0 \to \cdots$$

with $i = 0, 1$. Then the roofs $\mathcal{F}^\bullet \leftarrow \mathcal{F}^\bullet_0 \to \mathcal{H}^{m_0}[m_0]$ and $\mathcal{H}^{m_1}[m_1] \leftarrow \mathcal{F}^\bullet_1 \to \mathcal{F}^\bullet$ give morphisms in $\mathrm{D}^b(X)$ which induce isomorphisms in m_0^{th} and m_1^{th} cohomology respectively. Thus we get a non-trivial composition

$$\mathcal{F}^\bullet[m_0] \to \mathcal{H}^{m_0} \to \mathcal{H}^{m_1} \to \mathcal{F}^\bullet[m_1].$$

Hence, $m := m_0 = m_1$ and $\mathcal{F}^\bullet = \mathcal{F}[m]$ is a shifted coherent sheaf supported in x.

To conclude, recall that we have a non-trivial morphism $\mathcal{F} = \mathcal{H}^m \to \mathcal{H}^m$ given locally by $M \twoheadrightarrow A/\mathfrak{m} \hookrightarrow M$. Since $\mathcal{F}$ is simple, this morphism is invertible, in particular p and i are isomorphisms, and thus $M \simeq A/\mathfrak{m}$, *i.e.* $\mathcal{H}^m \simeq k(x)$. □

Proposition 7.5 (Bondal, Orlov)**.** *Let X be a smooth projective variety. Suppose that ω_X or ω_X^* is ample. Then the point like objects in $\mathrm{D}^b(X)$ are the objects which are isomorphic to $k(x)[m]$ for some $x \in X$ closed point and $m \in \mathbb{Z}$.*

Proof. It's clear that any object $k(x)[m]$ is point like (see Remark 7.3). Now assume that $P \in \mathrm{D}^b(X)$ satisfies conditions $1-3$ of Definition 7.2. Denote by $\mathcal{H}^i$ the cohomology of P. Then condition 1 ensures that $\mathcal{H}^i \otimes \omega_X[n] \simeq \mathcal{H}^i[d]$, hence $d = n$ and $\mathcal{H}^i \simeq \mathcal{H}^i \otimes \omega_X$.

Now, recall that the Hilbert polynomial associated to a coherent sheaf $\mathcal{F}$ is defined as

$$P_{\mathcal{F}}(t) = \chi(\mathcal{F} \otimes \omega_X^t),$$

where $\chi_{\mathcal{F}} := \sum_i \dim_k H^i(X, \mathcal{F})$ is the Euler-Poincaré characteristic (note that once again we use that all cohomology groups are finite dimensional). Now, we use that the degree of $P_{\mathcal{F}}$ is exactly the dimension of $\mathrm{Supp}(\mathcal{F})$ (see [14], II, 6, 81). Then, if $\dim(\mathrm{Supp}(\mathcal{F})) > 0$, we have that there exist non-zero integers p, q such that $\mathcal{F} \otimes \omega_X^p \not\simeq \mathcal{F} \otimes \omega_X^q$. Hence $\mathcal{F} \not\simeq \mathcal{F} \otimes \omega_X$.

We conculde that $\mathrm{Supp}(\mathcal{H}^i)$ is zero-dimensional for all i, and the assertion follows from Lemma 7.4 applied to P. □

Definition 7.6. Let $\mathcal{D}$ be a triangulated category with a Serre functor S. An object $L \in \mathcal{D}$ is called *invertible* if for any point like object $P \in \mathcal{D}$ there exists $n_P \in \mathbb{Z}$ such that

$$\mathrm{Hom}(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.7 (Bondal, Orlov)**.** *Let X be a smooth projective variety. Any invertible object in $\mathrm{D}^b(X)$ is of the form $L[m]$ with L a line bundle on X and $m \in \mathbb{Z}$. Conversely, if ω_X or ω_X^* is ample, then for any line bundle L and any $m \in \mathbb{Z}$ the object $L[m] \in \mathrm{D}^b(X)$ is invertible.*

Proof. Let L be an invertible object in $\mathrm{D}^b(X)$ and let m be maximal with $\mathcal{H}^m := \mathcal{H}^m(L) \not\simeq 0$. Then there exists a morphism

$$L \to \mathcal{H}^m[-m]$$

(see proof of Lemma 7.4). Fix a point $x_0 \in \mathcal{H}^m$. There exists a non-trivial morphism $\mathcal{H}^m \to k(x_0)$. Notice that

$$\mathrm{Hom}(\mathcal{H}^m, k(x_0)) \simeq \mathrm{Hom}(L, k(x_0)[-m]).$$

Indeed, any morphism $\varphi : \mathcal{H}^m \to k(x_0)$ can be composed with $L \to \mathcal{H}^m[-m]$ (up to shift φ by $[-m]$), and conversely any morphism $L \to k(x_0)[-m]$ is only determined by its restriction to $\mathcal{H}^m$. Thus we obtain

$$0 \not\simeq \mathrm{Hom}(L, k(x_0)[-m])$$

and therefore $n_{k(x_0)} = -m$.

Now consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^{-q}(L), k(x_0)[p]) \Rightarrow \mathrm{Hom}(L, k(x_0)[p+q]). \tag{4}$$

We deduce that

$$E_2^{1,-m} = \mathrm{Hom}(\mathcal{H}^m, k(x_0)[1]) = \mathrm{Hom}(L, k(x_0)[1 + n_{k(x_0)}]) = 0.$$

Thus $\mathrm{Ext}^1(\mathcal{H}^m, k(x_0)) = 0$ for any point $x_0 \in \mathrm{Supp}(\mathcal{H}^m)$. Now we want to apply the following lemma of commutative algebra.

Lemma 7.8. *Any finite module M over a noetherian local ring $(A, \mathfrak{m})$ with $\mathrm{Ext}^1(M, A/\mathfrak{m}) = 0$ is free.*

Proof. Set $k = A/\mathfrak{m}$. Any k-base of $M/\mathfrak{m}M$ lifts to a minimal set of A-generators of M by Nakayama lemma. We obtain the exact sequence of A-modules

$$0 \longrightarrow N \xrightarrow{\varphi} A^n \longrightarrow M \longrightarrow 0$$

and N is finitely generated since A is noetherian. Then φ induces the trivial morphism $\widetilde{\varphi} : N/\mathfrak{m}N \to k^n$. The vanishing of $\mathrm{Ext}^1(M, k)$ (seen as the derived functor of $\mathrm{Hom}_A(\ \ , k)$) leads to a surjective morphism

$$\mathrm{Hom}_A(A^n, k) \to \mathrm{Hom}(N, k).$$

Now, notice that $\mathrm{Hom}(A^n, k) \simeq \mathrm{Hom}_k(k^n, k)$ and $\mathrm{Hom}_A(N, k) \simeq \mathrm{Hom}_k(N/\mathfrak{m}N, k)$, and thus the morphism

$$\mathrm{Hom}_k(k^n, k) \to \mathrm{Hom}_k(N/\mathfrak{m}N, k)$$

given by $\widetilde{\varphi}$ is surjective. But since $\widetilde{\varphi} = 0$, this is only possible if $N/\mathfrak{m}N = 0$, thus $N = 0$ by Nakayama lemma. Then M is free. $\square$

Now we use the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m, k(x_0)).$$

Note that this spectral sequence exists since $R\Gamma \circ R\mathcal{H}om(\mathcal{F}^\bullet,\ \ ) = R\,\mathrm{Hom}(\mathcal{F}^\bullet,\ \ )$, but one has to check that the image of a complex of injectives under $\mathcal{H}om(\mathcal{F}^\bullet,\ \ )$ is Γ-acyclic. Since $k(x_0)$ is concentrated in one point, so is $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))$, and hence it is flasque. Thus we obtain

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0))) = 0$$

(see Remark 5.13). Moreover, $E_2^{-2,2} = 0$ since there is no negative cohomology, and hence $E_2^{0,1} = E_\infty^{0,1}$. But we know that $E^1 = 0$, thus $E_\infty^{0,1} = 0$. Now, $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0))$ is also concentrated in $\{x_0\}$ since an injective resolution of $k(x_0)$ can be chosen to be concentrated in $\{x_0\}$. In particular, it means that $\mathcal{E}xt^1(\mathcal{H}^m, x_0)$ is generated by global sections, and thus

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))) = 0$$

implies that $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0$. Finally, since $\mathcal{H}^m$ is coherent we can use ([8], III, Proposition 6.8) and we get

$$\mathrm{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m_{x_0}, k(x_0)) = \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))_{x_0} = 0.$$

Thus, we can apply Lemma 7.8 to conclude that $\mathcal{H}^m_{x_0}$ is free.

Note that freeness is an open property: pick $x_0 \in \mathrm{Supp}(\mathcal{H}^m)$ and consider an affine neighborhood $U = \mathrm{Spec}\, A \subseteq \mathrm{Supp}(\mathcal{H}^m)$. On U, the coherent sheaf $\mathcal{H}^m|_U$ correspond to a finitely generated A-module M. Since A is noetherian, M is also finitely presented. Thus we have an exact sequence

$$0 \to N \to A^{\oplus p} \to M \to 0.$$

It induces an exact sequence of localized modules

$$0 \to N_x \to A_x^{\oplus p} \to M_x \to 0.$$

Choose a minimal set of generators $a_1, \ldots, a_l$ of N. Since $\mathcal{H}_x^m$ is free, the a_i's restrict to 0 in N_x. Hence there exist open neighborhood U_i of a_i, $i = 1, \ldots, l$, such that $a_i = 0$. Since $x \in U_i$ for all i, $\cap U_i \neq \emptyset$ and we have that $N|_{\cap U_i} = 0$. Thus $\mathcal{H}^m$ is free on $\cap U_i$.

We have proved that $\mathrm{Supp}(\mathcal{H}^m)$ contains an open (dense) subset of X, but since $\mathcal{H}^m$ is coherent, its support is closed (Corollary 4.18). Since X is irreducible, we obtain $\mathrm{Supp}(\mathcal{H}^m) = X$ and $\mathcal{H}^m$ is locally free. Thereby, there exists for any $x \in X$ a surjection $\mathcal{H}^m \to k(x)$. Hence,

$$\mathrm{Hom}(L, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m, k(x)) \neq 0.$$

In particular, $n_{k(x)}$ does not depend on x. We have

$$\begin{aligned} k(x) &= \mathrm{Hom}(L, k(x)[-m]), \\ &= \mathrm{Hom}(\mathcal{H}^m, k(x)), \\ &= \mathrm{Hom}(\mathcal{O}_{X,x_0}^{\oplus p}, k(x)), \\ &\simeq k(x)^{\oplus p}. \end{aligned}$$

Hence we conclude that $p = 1$ and $\mathcal{H}^m$ is a line bundle.

Now we show that $\mathcal{H}^i = 0$ for $i < m$. Consider again the spectral sequence (4). We have

$$E^{q,-m} = \mathrm{Ext}^q(\mathcal{H}^m, k(x)) \simeq H^q(X, (\mathcal{H}^m)^\vee \otimes k(x)) = 0 \tag{5}$$

for $q > 0$ because $(\mathcal{H}^m)^\vee \otimes k(x)$ is supported in $\{x\}$ (skyscraper sheaf) and thus is flasque.

By induction, assume we have shown $\mathcal{H}^i = 0$ for $i_0 < i < m$. Then $E_2^{2,-i_0-1} = 0$ because either $i_0 < i_0 + 1 < m$ and the inductive hypothesis apply or $i_0 + 1 = m$ and the equality holds by (5). Moreover $E_2^{-2,-i_0+1} = 0$ because there is no negative Ext between sheaves. Thus $E_2^{0,-i_0} = E_\infty^{0,-i_0}$. Since $i_0 \neq m$ we have

$$E^{-i_0} = \mathrm{Hom}(L, k(x)[-i_0]) = 0,$$

thus $\mathrm{Hom}(\mathcal{H}^{i_0}, k(x)) = 0$ for all $x \in X$, *i.e.* $\mathcal{H}^{i_0} = 0$. As explained above, the induction starts with $i_0 = m - 1$.

Conversely, assume that the (anti)-canonical bundle is ample. We know by Proposition 7.5 that any point like object P in $\mathrm{D}^b(X)$ is of the form $k(x)[l]$ for some closed point $x \in X$ and some $l \in \mathbb{Z}$. Hence

$$\begin{aligned} \mathrm{Hom}(L[m], P[i]) &= \mathrm{Hom}(L, k(x)[l + i - m]), \\ &\simeq \mathrm{Ext}^{i+l-m}(L, k(x)), \\ &\simeq \mathrm{Ext}^{i+l-m}(\mathcal{O}_X, L^\vee \otimes k(x)), \\ &\simeq H^{i+l-m}(X, L^\vee \otimes k(x)) = 0 \end{aligned}$$

except for $i = m - l$ because $L^\vee \otimes k(x)$ is flasque. We set $n_P := m - l$.

□

Theorem 7.9 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties and assume that ω_X or ω_X^* is ample. If there exists an exact equivalence $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$, then X and Y are isomorphic. In particular, the (anti)-canonical bundle of Y is also ample.*

Proof. We will proceed in several steps. Notice that since the notions of point like objects and invertible objects in $\mathrm{D}^b(X)$ are intrinsic, the equivalence $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ induces bijections:

$$\{\text{point like objects in } \mathrm{D}^b(X)\} \xleftrightarrow{\varphi} \{\text{point like objects in } \mathrm{D}^b(Y)\},$$

$$\{\text{invertible objects in } \mathrm{D}^b(X)\} \xleftrightarrow{\psi} \{\text{invertible objects in } \mathrm{D}^b(Y)\}.$$

Step 1 : $F(\mathcal{O}_X) = \mathcal{O}_Y$.

By Proposition 7.7, we know that $\mathcal{O}_X$ is an invertible object in $\mathrm{D}^b(X)$. Then $F(\mathcal{O}_X)$ must be an invertible object in $\mathrm{D}^b(Y)$, and then Proposition 7.7 implies that $F(\mathcal{O}_X) = M[m]$ for some line bundle M on Y and some integer $m \in \mathbb{Z}$. Hence, composing F with the two equivalences $M^* \otimes (\ )$ and $(\ )[-m]$ we obtain a new equivalence, which we continue to call F, that satisfies $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Step 2 : $F(\omega_X^k) = \omega_Y^k$ for all $k \in \mathbb{Z}$.

Denote by S_X (resp. S_Y) the Serre functor on X (resp. Y) given by Theorem 6.8. We know that $n = \dim X = \dim Y$ by Proposition 7.1, and that F commutes with the Serre functors by Lemma 1.5. Hence for any $k \in \mathbb{Z}$ we have:

$$\begin{aligned} F(\omega_X^k) &= F(S_X^k(\mathcal{O}_X))[-kn], \\ &\simeq S_Y^k(F(\mathcal{O}_X))[-kn], \\ &\simeq S_Y^k(\mathcal{O}_Y)[-kn], \\ &= \omega_Y^k. \end{aligned}$$

Step 3 : $\bigoplus H^0(X, \omega_X^k) \simeq \bigoplus H^0(Y, \omega_Y^k)$.

Using that F is fully faithful and the previous steps, we get

$$\begin{aligned} H^0(X, \omega_X^k) &\simeq \mathrm{Hom}(\mathcal{O}_X, \omega_X^k), \\ &\simeq \mathrm{Hom}(\mathcal{F}(\mathcal{O}_X), F(\omega_X^k)), \\ &\simeq \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k), \\ &\simeq H^0(Y, \omega_Y^k). \end{aligned}$$

Now, the product in $\bigoplus H^0(X, \omega_X^k)$ can be expressed as follows. Take $s_i \in H^0(X, \omega_X^{k_i})$, $i = 1, 2$. Since S_X is an equivalence, we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_X, \omega_X^{k_2}) &\simeq \mathrm{Hom}(S_X^{k_1}(\mathcal{O}_X), S_X^{k_1}(\omega_X^{k_2})), \\ &\simeq \mathrm{Hom}(\omega_X^{k_1}[k_1 n], \omega_X^{k_1+k_2}[k_1 n]), \\ &\simeq \mathrm{Hom}(\omega_X^{k_1}, \omega_X^{k_1+k_2}). \end{aligned}$$

Thus, the product $s_1 \cdot s_2$ is given by $\widetilde{s_2} \circ s_1$, where $\widetilde{s_2} : \omega_X^{k_1} \to \omega_X^{k_1+k_2}$ is given by the isomorphism above. It is easy to see that $F(s_1 \cdot s_2) = F(s_1) \cdot F(s_2)$ since F commute with shifts and Serre functors. Thus we conclude that the ring isomorphism

$$\bigoplus H^0(X, \omega_X^k) \simeq \bigoplus H^0(Y, \omega_Y^k)$$

is a graded ring isomorphism.

Step 4 : The (anti)-canonical bundle of Y is ample.

This is the longest step. In order to prove it, we shall first prove that point like objects in $\mathrm{D}^b(Y)$ are of the form $k(y)[m]$. For any closed point $y \in Y$, denote by $x_y \in X$ the closed point verifying $F(k(x_y)[m_y]) \simeq k(y)$, $m_y \in \mathbb{Z}$, induced by φ. Suppose that $P \in \mathrm{D}^b(Y)$ is a point like object which is not of the form $k(y)[m]$ and denote by $x_P \in X$ the closed point with $F(k(x_P)[m_P]) \simeq P$. Note that $x_P \neq x_y$ for all $y \in Y$, hence we have for all $y \in Y$ and all $m \in \mathbb{Z}$

$$\begin{aligned} \mathrm{Hom}(P, k(y)[m]) &\simeq \mathrm{Hom}(F(k(x_P))[m_P], F(k(x_y))[m_y + m]), \\ &\simeq \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) = 0. \end{aligned}$$

Indeed, $\mathrm{Supp}(k(x_P)) \cap \mathrm{Supp}(k(x_y)) = \emptyset$ so there is no Ext between them. But objects of the form $k(y)$ form a spanning class in $\mathrm{D}^b(Y)$ (Proposition 6.11) thus $P \simeq 0$ which is absurd (indeed, a point like object cannot be trivial since its endomorphisms form a *field*, so in particular $0 \neq \mathrm{Id}$). Thus point like objects in $\mathrm{D}^b(Y)$ are exactly the objects of the form $k(y)[m]$.

Note that for any closed point $x \in X$ there exists a closed point $y \in Y$ such that $F(k(x)) \simeq k(y)$ (no shifts). Indeed, $0 \neq \mathrm{Hom}(\mathcal{O}_X, k(x)) \simeq \mathrm{Hom}(F(\mathcal{O}_X), F(k(x)))$ since F is fully faithful, but $\mathrm{Hom}(\mathcal{O}_Y, k(y)[m]) \simeq H^m(Y, k(y))$ and the latter is non-zero if and only if $m = 0$ because $k(y)$ is flasque.

Finally, it remains to prove that ω_Y^k is very ample if ω_X^k is so. To do so we will use Proposition 5.6 (here we use that k is algebraically closed, but this hypothesis can be omited, see [1]).

The line bundle ω_Y^k separates points if for any two points $y_1 \neq y_2$ of Y the restriction map

$$r : H^0(Y, \omega_Y^k) \to H^0(Y, k(y_1) \oplus k(y_2))$$

is surjective (Remark 5.8). Denote by x_i, $i = 1, 2$, the closed point x_{y_i}, where we use the notation of the previous paragraph. The restriction map $\omega_Y^k \to k(y_1) \oplus k(y_2)$ is sent through the isomorphisms

$$\begin{aligned} \mathrm{Hom}(\omega_Y^k, k(y_1) \oplus k(y_2)) &\simeq \mathrm{Hom}(F(\omega_Y^k), F(k(y_1) \oplus k(y_2))) \\ &\simeq \mathrm{Hom}(\omega_X^k, k(x_1) \oplus k(x_2)) \end{aligned}$$

to the restriction map $\omega_X^k \to k(x_1) \oplus k(x_2)$ since there is only one non-trivial morphism $\omega_X^k \to k(x_i)$ up to scaling. We obtain the commutative diagram

$$\begin{tikzcd} H^0(Y, \omega_Y^k) \arrow[r] \arrow[d, "\sim"] & H^0(Y, k(y_1) \oplus k(y_2)), \arrow[d, "\sim"] \\ \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(\mathcal{O}_Y, k(y_1) \oplus k(y_2)), \arrow[d, "\sim"] \\ \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(\mathcal{O}_X, k(x_1) \oplus k(x_2)), \arrow[d, "\sim"] \\ H^0(X, \omega_X^k) \arrow[r, two heads] & H^0(X, k(x_1) \oplus k(x_2)). \end{tikzcd}$$

Since the map $H^0(X, \omega_X^k) \to H^0(X, k(x_1) \oplus k(x_2))$ is onto (because ω_X^k separates points), so is the map $H^0(Y, \omega_Y^k) \to H^0(Y, k(y_1) \oplus k(y_2))$.

Now, we need to show that ω_Y^k separates tangent directions. We will use again Remark 5.8. Assume that $Z_y \subseteq Y$ is a subscheme of length two concentrated in $y \in Y$, that is y endowed with a tangent direction (see Proposition 4.26). We have an exact sequence

$$0 \to k(y) \to \mathcal{O}_{Z_y} \to k(y) \to 0$$

given by $k(y) \ni \lambda \mapsto \lambda \cdot m \in \mathcal{O}_{Z_y}$ for a k-generator m of $\mathfrak{m}_{Z_y}$ and $\mathcal{O}_{Z_y} \ni a \mapsto [a] \in \mathcal{O}_{Z_y}/\mathfrak{m}_{Z_y} \simeq k$. Such a (non-trivial) extension is given by an element

$$e_Z \in \mathrm{Hom}(k(y), k(y)[1]) = \mathrm{Hom}(F(k(x)), F(k(x))[1]) = \mathrm{Hom}(k(x), k(x)[1])$$

(see Remark 2.10).

The latter, when viewed as a class in $\mathrm{Hom}(k(x), k(x)[1])$, defines an extension of $\mathcal{O}_{X,x}$-modules

$$0 \to k(x) \to M \to k(x) \to 0.$$

It's in particular an extension of k-vector spaces, and thus we can endow M with a structure of k-algebra pulling back the structure of k-algebra of $k[T]/T^2$ through the k-linear isomorphism

$$M \xrightarrow{\sim} k[T]/T^2$$

given by choising a k-basis of M compatible with the extension. Thus $\mathcal{O}_{Z_x} := M$ define a subscheme $Z_x \subseteq X$ of length two concentrated in x and $F(\mathcal{O}_{Z_x}) = \mathcal{O}_{Z_y}$. Since ω_X^k separates tangent directions, the restriction

$$H^0(X, \omega_X^k) \to H^0(X, \mathcal{O}_{Z_x})$$

is surjective. Now using $H^0(X, \omega_X^k) = H^0(Y, \omega_Y^k)$ and

$$\begin{aligned} H^0(Y, \mathcal{O}_{Z_y}) &\simeq \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_{Z_y}), \\ &\simeq \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_{Z_y}), \\ &\simeq H^0(X, \mathcal{O}_{Z_x}) \end{aligned}$$

we deduce the surjectivity of $H^0(Y, \omega_Y^k) \to H^0(Y, \mathcal{O}_{Z_y})$, *i.e.* ω_Y^k separates the tangent direction in y given by Z_y.

Step 5 : End of the proof.

We proved step 3 that $\bigoplus H^0(X, \omega_X^k) \simeq \bigoplus H^0(Y, \omega_Y^k)$. Using that both (anti)-canonical bundle of X and Y are ample, we conclude using Theorem 5.19:

$$X \simeq \mathrm{Proj} \bigoplus H^0(X, \omega_X^k) \simeq \mathrm{Proj} \bigoplus H^0(Y, \omega_Y^k) \simeq Y.$$

□

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