Hilbert uniformization of Riemann surfaces : I

SHORT VERSION

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Abstract

We report on a cell decompostion for the moduli space of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ boundary curves and $m \geq 0$ punctures.

1 Moduli spaces and mapping class groups

Let $\mathfrak{M} = \mathfrak{M}^m_{g,n}$ denote the moduli space of conformal equivalence classes of Riemann surfaces $F = F^m_{g,n}$ of genus $g \geq 0$ and with $n \geq 1$ boundary curves and $m \geq 0$ permutable punctures. Likewise, let $\Gamma = \Gamma^m_{g,n}$ be the corresponding mapping class group of isotopy classes of orientation-preserving diffeomormphisms fixing the boundary pointwise and permute the punctures.

Since $n \geq 1$, the automorphisms of F are trivial, and thus the action of Γ on the corresponding Teichmüller space is free. Therefore the moduli space $\mathfrak{M}$ is a smooth, non-compact manifold, with the homotopy type of the classifying space $B\Gamma$. Its dimension is $d = 6g - 6 + 3n + 2m$. Because we allow punctures to be permuted, it is orientable only in the cases $m = 0$ or $m = 1$.

2 Results

There is a flat vector bundle $\mathfrak{D} \to \mathfrak{M}$ of dimension $\bar{d} = m + 3n$, the fibres of which are vector spaces of certain harmonic functions. Following an earlier version [Bödigheimer-90], we construct in [Bödigheimer-05] a finite cell complex $P = P(h, m, n)$ as a compactification of $\mathfrak{D}$; here $h = 2g + m + 2n - 2$ and $d + \bar{d} = 3h$. A point in P is a configuration of h pairs of horizontal, semi-infinite slits in n complex planes.

Theorem.
There is a subcomplex $P' \subset P$ and a homeomorphism $\mathcal{H} : \mathfrak{D} \longrightarrow P - P'$.

The inverse homeomorphism is defined and studied in detail [Bödigheimer-05]; the continuity of $\mathcal{H}$ is discussed in [Ebert-05].

3 Cell structure

To describe the space P we concentrate on the case of a single boundary curve. Fix g and m, and put $n = 1$; then $h = 2g + m$.

The space $P = P(h, m, n)$ is bi-simplicial complex $P_{p,q}$, where $0 \leq p \leq 2h$ and $0 \leq q \leq h$. The cells in $P_{p,q}$ are products $\Delta^q \times \Delta^p$ of simplices, and hey are given by q-tuples $\Sigma = (\sigma_q, \ldots, \sigma_0)$ of permutations σ_i in the symmetric group $\mathfrak{S}_{p+1}$, acting on $0, 1, \ldots, p$, satisfying the following conditions :

$$\text{norm}(\sigma) \leq h \tag{3.1}$$
$$\sigma_q \text{ has at most } m+1 \text{ cycles} \tag{3.2}$$

Here the norm(Σ) is the sum of the word lengths of all $\tau_i = \sigma_i \, \sigma_{i-1}^{-1}$ for $i = 1, \ldots, q$, measured with respect to the generating set of all transpositions.
The face operators are given by

$$d_i'(\Sigma) = (\sigma_q, \ldots, \widehat{\sigma_i}, \ldots, \sigma_0) \tag{3.3}$$

and

$$d_j''(\Sigma) = (D_j(\sigma_q), \ldots, D_j(\sigma_0)) \tag{3.4}$$

where $D_j : \mathfrak{S}_{p+1} \to \mathfrak{S}_p$ deletes the letter j from the cycle it occurs in and re-normalizes the indices.
The subcomplex P' consists of all Σ with norm less than h or with a σ_q having less than $m+1$ cycles, or where any of the following conditions is violated:

$$\sigma_i(p) = 0 \quad \text{for} \quad i = 0, \ldots, q \tag{3.5}$$
$$\sigma_0 \quad \text{is the rotation} \quad 0 \mapsto 1 \mapsto 2 \mapsto \ldots \mapsto p \mapsto 0 \tag{3.6}$$
$$\sigma_{i+1} \neq \sigma_i \quad \text{for} \quad i = 0, \ldots q-1 \tag{3.7}$$
$$\text{There is no} \quad k \in \{0, \ldots, p-1\} \quad \text{such that} \quad \sigma_i(k) = k+1 \quad \text{for all} \quad i = 0, \ldots q \tag{3.8}$$

Remarks.
The Hilbert unifomization method goes back to work of Hilbert and Courant. It can also be used to parametrize the moduli spaces of surfaces with incoming/outgoing boundary curves, see [Bödigheimer-03]; and it can be used for moduli spaces of conformal equivalence classes of non-orientable surfaces (Kleinian surfaces); see [Ebert-03], [Zaw-04].

References

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